28-th All-Russian Mathematical Olympiad 2002

Fourth Round

Grade 8

First Day

1. Can a natural number be written in each square of a 9×2002 rectangular board in such a way that the sum of the numbers in each row or column is a prime number?

2. Each cell of a 9×9 square board is colored red or blue. Show that there is a square having exactly two red diagonally adjacent squares or exactly two blue diagonally adjacent squares.

3. We are given 11 empty boxes. In a move one can choose 10 boxes and place a coin in each of them. Two players make moves alternately. The winner is the one after whose move there are 21 coins in one of the boxes. Who has a winning strategy?

4. Regular triangles ABC_1, BCA_1, CAB_1 are constructed externally on the sides of a scalene triangle ABC. Prove that the triangle $A_1B_1C_1$ cannot be regular.

Second Day

5. Given a four-digit natural number, one can increase two adjacent digits by 1 if none of them is 9, or decrease two adjacent digits by 1 if none of them is 0. Is it possible to turn number 1234 into number 2002 by applying such operations?

6. Each side of a convex quadrilateral is prolonged on both sides, all the eight prolongations being segments of equal length. Suppose that the second endpoints of these eight segments lie on a circle. Prove that the initial quadrilateral is a square.

7. An observer at a highway sees three cars moving with a constant speed: a Moskwich and a Zaporozhets, and a Niva moving towards them. When the Moskwich passes by the observer, it is equidistant from the other two cars; and when the Niva passes by the observer, it is equidistant from the other two cars. Show that, when Zaporozhets passes by the observer, it is also equidistant from the other two cars. (At the considered moments, the two equidistant cars are on the opposite sides of the observer.)

8. Among 18 pieces, some three weigh 99g each, while the others weigh 100g each. Determine the lighter pieces in two measurings on a balance with a scale.

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Grade 9

First Day

1. *Problem 2 for Grade 8.*

2. A monic quadratic trinomial with integer coefficients assumes prime values at three successive integer points. Prove that it assumes a prime value in at least one other integer point.

3. In a triangle ABC with $AB = BC$, O is the circumcenter, M a point on segment BO, and M' the point symmetric to M with respect to the midpoint of AB. Lines $M'O$ and AB meet at K. Point L on side BC is such that $\angle CLO = \angle BLM$. Prove that the points O,K,B,L are on a circle.

4. On the plane are positioned $[4n/3]$ rectangles with sides parallel to the coordinate axes. Assume that every rectangle intersects at least n other rectangles. Show that there exists a rectangle that intersects all other rectangles.

Second Day

5. Is it possible to arrange numbers $1,2,\dots,60$ on a circumference in such a way that the sum of any two numbers separated by one number is divisible by 2, the sum of any two numbers separated by two numbers is divisible by 3, etc, the sum of any two numbers separated by six numbers is divisible by 7?

6. Point A' on the boundary of a trapezoid $ABCD$ is such that the line AA' bisects the area of the trapezoid. Points B',C',D' are analogously defined. Prove that the intersection points of the diagonals of the quadrilaterals $ABCD$ and $A'B'C'D'$ are symmetric with respect to the midpoint of the middle line of the trapezoid $ABCD$.

7. On the segment $[0,2002]$, the endpoints and an interior point with an integer coordinate d coprime to 1001 are marked. We are allowed to mark the midpoint of a segment whose endpoints are marked, provided that its coordinate is integral. Can we mark all integral points on the segment after several such operations?

8. *Problem 8 for Grade 8.*

Grade 10

First Day

1. What is the largest length of an arithmetic progression of natural numbers $a_1,\dots,a_n$ with the common difference 2 having the property that a_k^2+1 is prime for all k?

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2. A convex polygon on the coordinate plane contains at least m^2+1 points with integer coordinates in its interior. Show that some $m+1$ of these points lie on a line.

3. The perpendicular bisector of side AC of a triangle ABC meets side BC at M. The bisecting ray of angle AMB meets the circumcircle of $\triangle ABC$ at K. Prove that the line passing through the incenters of triangles AKM and BKM is perpendicular to the bisector of angle AKB.

4. A sequence (a_n) of numbers satisfies $a_0 = 0$ and $0 \le a_{k+1} - a_k \le 1$. Prove the inequality
$$\sum_{k=0}^{n} a_k^3 \le \left(\sum_{k=0}^{n} a_k \right)^2 .$$

Second Day

5. On the axis Ox are arbitrarily taken different points $X_1, \dots, x_n$, $n \ge 3$. Consider all parabolas defined by monic quadratic trinomials and intersecting the axis Ox in the given points only. Let $y = f_1, \dots, y = f_m$ be the functions defining these parabolas. Prove that the parabola $y = f_1 + \dots + f_m$ intersects the axis Ox in two points.

6. *Problem 6 for Grade 9.*

7. On the segment $[0, 2002]$, the endpoints and $n-1 > 0$ interior points with integer coordinates are marked so that the lengths of the segments into which the segment $[0, 2002]$ is divided have the greatest common divisor 1. We are allowed to divide a segment with marked endpoints into n equal parts and to mark the division points, provided that their coordinates are integral. Can we mark all integral points on the segment after several such operations?

8. What maximal number of colors can we use to color the squares of a 10×10 square board so that each row or column contains squares of at most five different colors?

Grade 11

First Day

1. Real numbers x and y are such that $x^p + y^q$ is rational for any different odd primes p, q. Show that x and y are rational.

2. The altitude of a quadrilateral pyramid $SABCD$ passes through the intersection of the diagonals of the base $ABCD$. The perpendiculars AA_1, BB_1, CC_1, DD_1 are dropped onto the edges SC, SD, SA, SB, respectively. Suppose that the points S, A_1, B_1, C_1, D_1 are distinct and lie on a sphere. Show that the lines AA_1, BB_1, CC_1, DD_1 pass through a single point.

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3. A sequence (a_n) of numbers satisfies $a_0 = 0$ and $a_{k+1} \geq a_k + 1$. Prove the inequality
$$\sum_{k=1}^{n} a_k^3 \geq \left(\sum_{k=1}^{n} a_k\right)^2.$$

4. An infinite square grid is colored with n^2 colors so that all unit squares in any $n \times n$ square are of different colors. It is known that any infinite row contains all the colors. Show that there is an infinite column containing exactly n colors.

Second Day

5. Let $P(x)$ be a polynomial of an odd degree. Prove that the equation $P(P(x)) = 0$ has at least as many different real roots as the equation $P(x) = 0$ does.

6. On the plane are given $n > 1$ points (in a general position). Two players alternately connect two non-connected points by an arbitrarily oriented vector. The second player wins if, after several moves, all drawn vectors sum up to 0; otherwise, the first player wins. Who can force a victory?

7. Let l_A, l_B, l_C, l_D be the outer bisectors of the angles of a convex quadrilateral $ABCD$. The lines l_A and l_B meet at K, l_B and l_C at L, l_C and l_D at M, and l_D and l_A at N. Prove that if the circumcircles of triangles ABK and CDM are externally tangent, then so are the circumcircles of triangles BCL and DAN.

8. On the segment $[0,N]$, the endpoints and two more points are marked so that the lengths of the three segments into which the segment $[0,N]$ is divided are integers with the greatest common divisor 1. If A and B are two marked points whose distance is a multiple of 3, then we can divide the segment AB into three equal parts, mark one of the division points and unmark one of the points A,B. Is it true that we can mark any given integer point on the segment AB with several such operations?

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