34-th All-Russian Mathematical Olympiad 2008

Final Round – Kislovodsk, April 19–24

Grade 9

First Day

1. Do there exist 14 positive integers such that, upon increasing each of them by 1, their product increases exactly 2008 times?

2. Numbers a,b,c are such that the equation $x^3+ax^2+bx+c=0$ has three real roots. Prove that if $-2 \leq a+b+c \leq 0$ then at least one of these roots belongs to the segment $[0,2]$.

3. In a scalene triangle ABC, H and M are the orthocenter an centroid respectively. Consider the triangle formed by the lines through A,B and C perpendicular to AM,BM and CM respectively. Prove that the centroid of this triangle lies on the line MH.

4. There are several scientists collaborating in Niichavo. During an 8-hour working day, the scientists went to the cafeteria, possibly several times. It is known that for every two scientists, the total time in which exactly one of them was in cafeteria is at least x hours ($x > 4$). What is the largest possible number of scientists that could work in Niichavo that day, in terms of x?

   Remark. Niichavo is an institute of wizardry from a Russian novel.

Second Day

5. The *distance* between two cells of an infinite chessboard is defined as the minimal number of moves needed for a kind to move from one to the other. On the board are chosen three cells on the pairwise distances equal to 100. How many cells are there that are on the distance 50 from each of the three cells?

6. The incircle of a triangle ABC touches the side AB at X and AC at Y. Let K be the midpoint of the arc AB of the circumcircle of $\triangle ABC$. Assume that the line XY bisects the segment AK. What are the possible measures of angle BAC?

7. A natural number is written on the blackboard. Whenever number x is written on the blackboard, one can write any of the numbers $2x+1$ and $\frac{x}{x+2}$. At some moment the number 2008 was noticed on the blackboard. Show that it was there from the very beginning.

8. We are given 3^{2k} apparently identical coins, one of which is fake, being lighter than the others. We also dispose of three apparently identical balances without weights, one of which is broken (and yields outcomes unrelated to the actual situations). How can we find the fake coin in $3k+1$ weighings?

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Grade 10

First Day

1. *Problem 1 for Grade 9.*

2. The columns of a given $n \times n$ board are labeled 1 to n. The numbers $1 \dots, n$ are arranged in the cells of the board so that the numbers in each row or column are pairwise different. We call a cell *good* if the number in it is greater than the label of its column. For which n is there an arrangement in which each row contains equally many good cells?

3. A circle ω with center O is tangent to the rays of an angle BAC at B and C. Point Q is taken inside the angle BAC. Assume that point P on the segment AQ is such that $AQ \perp OP$. The line OP intersects the circumcircles ω_1 and ω_2 of triangles BPQ and CPQ again at points M and N. Prove that $OM = ON$.

4. The sequences (a_n) and (b_n) are defined by $a_1 = 1$, $b_1 = 2$ and
$$a_{n+1} = \frac{1 + a_n + a_n b_n}{b_n}, \quad b_{n+1} = \frac{1 + b_n + a_n b_n}{a_n}.$$
Show that $a_{2008} < 5$.

Second Day

5. Determine all triples of real numbers x, y, z satisfying
$$1 + x^4 \leq 2(y - z)^2, \quad 1 + y^4 \leq 2(z - x)^2, \quad 1 + z^4 \leq 2(x - y)^2.$$

6. In a scalene triangle ABC the altitudes AA_1 and CC_1 intersect at H, O is the circumcenter, and B_0 the midpoint of side AC. The line BO intersects side AC at P, while the lines BH and A_1C_1 meet at Q. Prove that the lines HB_0 and PQ are parallel.

7. For which integers $n > 1$ do there exist natural numbers $b_1, \dots, b_n$, not all equal, such that the number $(b_1 + k)(b_2 + k) \cdots (b_n + k)$ is a power of an integer for each natural number k? (The exponents may depend on k, but must be greater than 1.)

8. On the cartesian plane are drawn several rectangles with the sides parallel to the coordinate axes. Assume that any two rectangles can be cut by a vertical or horizontal line. Show that it is possible to draw one horizontal and one vertical line such that each rectangle is cut by at least one of these two lines.

Grade 11

First Day

1. *Problem 2 for Grade 9.*

2. Petya and Vasya are given equal sets of N weights, in which the masses of any two weights are in ratio at most 1.25. Petya succeeded to divide his set into 10 groups of equal masses, while Vasya succeeded to divide his set into 11 groups of equal masses. Find the smallest possible N.

3. Given a finite set P of prime numbers, prove that there exists a positive integer x which is representable in the form $x = a^p + b^p$ (with $a, b \in \mathbb{N}$) for each $p \in P$, but not representable in that form for any $p \notin P$.

4. Each face of a tetrahedron can be placed in a circle of radius 1. Show that the tetrahedron can be placed in a sphere of radius $\frac{3}{2\sqrt{2}}$.

Second Day

5. The numbers from 51 to 150 are arranged in a 10×10 array. Can this be done in such a way that, for any two horizontally or vertically adjacent numbers a and b, at least one of the equations $x^2 - ax + b = 0$ and $x^2 - bx + a = 0$ has two integral roots?

6. A magician should determine the area of a hidden convex 2008-gon $A_1A_2 \dots A_{2008}$. In each step he chooses two points on the perimeter, whereas the chosen points can be vertices or points dividing selected sides in selected ratios. Then his helper divides the polygon into two parts by the line through these two points and announces the area of the smaller of the two parts. Show that the magician can find the area of the polygon in 2006 steps.

7. In a convex quadrilateral $ABCD$, the rays BA and CD meet at P, the rays BC and AD meet at Q, and D is the projection of D on PQ. Prove that the quadrilateral $ABCD$ is tangent if and only if the incircles of triangles ADP and CDQ are visible from H under the same angle.

8. On a chess tournament $2n + 3$ players take part. Every two play exactly one match. The schedule of the tournament has been made so that the matches are played one after another, and each player, after playing a match, is free in at least n consequent matches. Prove that one of the players who played the opening match on the tournament will also play the closing match.