33-rd All-Russian Mathematical Olympiad 2007

Final Round – Maykop, April 23–28

Grade 8

First Day

1. If a, b, c are real numbers, show that at least one of the equations $x^2+(a-b)x+(b-c)=0$, $x^2+(b-c)x+(c-a)=0$, $x^2+(c-a)x+(a-b)=0$ has a real solution.

2. Natural numbers from 1 to 100 are arranged in a 10×10 board. In each step it is allowed to exchange places of two numbers. Show that one can always perform 35 steps so that in the resulting board the sum of any two numbers adjacent by side is a composite number.

3. On side BC of a rhombus $ABCD$ is taken a point M. The lines through M perpendicular to diagonals BD and AC intersect side AD at points P and Q respectively. Assume that the lines PB, QC and AM have a common point. What are the possible values of the ratio BM/MC?

4. Magician Arutyun and his assistant Amayak perform the following trick. A circle has been drawn on the board. While the magician is away, an onlooker marks 2007 points on the circle and the assistant erases one of them. Then the magician enters the room, looks at the picture and determines a semicircle on which the erased point was lying. How can the magician make a deal with the assistant so that the trick is always successful?

Second Day

5. The distance between Maykop and Belorechensk is 24km. There are three friends, two of which are going from Maykop to Belorechensk and the third from Belorechensk to Maykop. They have one bicycle, initially located in Maykop. Each friend can go either by walking at the velocity of 6km/h or by bicycle at 18km/h. They do not leave the bicycle unattended and no two of them can use it at the same time. Show that they can all reach their destinations in 2 hours and 40 minutes.

6. A line through the incenter I of a triangle ABC meets the sides AB and BC at points M and N respectively so that triangle BMN is acute. Points K and L are taken on side AC so that $\angle ILA = \angle IMB$ and $\angle IKC = \angle INB$. Prove that $AM + KL + CN = AC$.

7. For an integer $n > 3$ denote by $n?$ the product of all prime numbers less than n. Solve the equation $n? = 2n + 16$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

8. The numbers from 1 to 100 are arranged in a 10×10 table: in the first row 1 through 10 from left to right, in the second row 11 through 20 from left to right, etc. Andrey wants to cut the table into 1×2 rectangles, compute the product of numbers in each rectangle and sum these products up. How should he cut the table in order to get the least possible sum?

Grade 9

First Day

1. Monic quadratic trinomials $f(x)$ and $g(x)$ are such that the equations $f(g(x)) = 0$ and $g(f(x)) = 0$ have no real solutions. Prove that at least one of the equations $f(f(x)) = 0$ and $g(g(x)) = 0$ also has no real solutions.

2. A hundred fractions are written, whose numerators are all integers from 1 to 100 and whose denominators are also all integers from 1 to 100. Suppose that the sum of these fractions is an irreducible fraction with the denominator 2. Prove that it is possible to exchange the numerators of two fractions so that the sum of the 100 fractions becomes an irreducible fraction with an odd denominator.

3. Two players alternately draw diagonals of a regular $(2n+1)$-gon ($n > 1$). Drawing a diagonal is allowed if the it intersects (at interior points) an even number of previously drawn diagonals (and has not been drawn before). The player who cannot make a move loses. Who wins if both players play intelligently?

4. In a triangle ABC, BB_1 is an angle bisector $(B_1 \in AC)$ and the perpendicular from B_1 to BC meets the arc BC of the circumcircle of $\triangle ABC$ at K. The perpendicular from B to AK meets AC at L. Prove that the points K, L and the midpoint of the arc AC (not containing B) lie on a line.

Second Day

5. At each vertex of a convex 100-gon are written two different numbers. Prove that it is always possible to erase a number from each vertex so that the remaining numbers at any two adjacent vertices are distinct.

6. In an acute-angled triangle ABC, M and N are the midpoints of AB and BC respectively and H the feet of the altitude from B. The circumcircles of triangles AHN and CHM intersect at point $P \neq H$. Prove that the line PH bisects the segment MN.

7. *Problem 8 for Grade 8.*

8. Dima wrote the reciprocals of the factorials of the natural numbers from 80 to 99 on 20 infinite paper stripes (for instance, on the last stripe he wrote $\frac{1}{99!} = 0.00\dots0010715\dots$ with 155 zeros after the decimal point). Sasha wants to cut a portion of one of the stripes containing N consecutive digits and no decimal

point. For which largest N can he do this so that Dima cannot determine the stripe which the portion was cut from?

Grade 10

First Day

1. The faces of a cube of edge 9 are partitioned into unit squares. The surface of the cube is covered without overlapping by 2×1 paper tiles, each covering exactly two unit squares. Prove that the number of folded tiles is even.
2. Consider a polynomial $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$. Denote $m = \min\{a_0, a_0 + a_1, \ldots, a_0 + a_1 + \cdots + a_n\}$. Prove that $P(x) \geq mx^n$ for $x \geq 1$.
3. *Problem 4 for Grade 9.*
4. A magician with his assistant performs the following trick. An onlooker writes a sequence of N digits on the board. The assistant covers some two adjacent digits with a black disk. Then the magician comes and attempts to guess the two covered digits (in the correct order). For which smallest N can the magician make a deal with the assistant so that the trick is always successful?

Second Day

5. A set of $n > 2$ vectors is given. A vector from the set is called *lengthy* if its length is not less than that of the sum of the other vectors. Prove that if each vector from the set if lengthy, then the sum of these vectors is zero.
6. Two circles ω_1 and ω_2 intersect at points A and B. Let PQ and RS be the common tangents to the circles, with P, R on ω_1 and Q, S on ω_2. Assume that $RB \parallel PQ$. The ray RB meets ω_2 again at W. Find the ratio RB/BW.
7. In a convex polyhedron, vertex A has degree 5, while all other vertices have degree 3 (the degree of a vertex is the number of edges meeting at it). A coloring of the edges of the polyhedron is said to be *good* if all edges at a vertex of degree 3 have different colors. Suppose that the number of good colorings is not divisible by 5. Prove that there is a good coloring in which at least three edges at A have the same color.
8. *Problem 8 for Grade 9.*

Grade 11

First Day

1. Show that for $k > 10$ one can replace one cos sign with sin in the product

$$f(x) = \cos x \cos 2x \cos 3x \cdots \cos 2^k x$$

in such a way that the obtained function f_1 satisfies $|f_1(x)| \le \frac{3}{2^{k+1}}$ for all real x.

2. The incircle of triangle ABC touches the sides BC, CA, AB at points A_1, B_1 and C_1 respectively. The segment AA_1 intersects the incircle again at point Q. The line l through A parallel to BC meets the lines A_1C_1 and A_1B_1 at P and R respectively. Prove that $\angle PQR = \angle B_1QC_1$.

3. *Problem 4 for Grade 10.*

4. In a sequence $(x_n)_{n=1}^{\infty}$ the first term x_1 is a rational number greater than 1 and $x_{n+1} = x_n + \frac{1}{[x_n]}$ for all natural n. Prove that this sequence contains an integer.

Second Day

5. *Problem 5 for Grade 9.*

6. Do there exist nonzero numbers a, b, c such that for each $n > 3$ there is a polynomial of the form $P_n(x) = x^n + \cdots + ax^2 + bx + c$ having exactly n integral roots (not necessarily distinct)?

7. Given a tetrahedron, Lesha wants to construct two spheres with two opposite edges of the tetrahedron as diameters. Can he always do this in such a way that the two spheres cover the entire tetrahedron?

8. There are N cities in a country, some pairs of which are connected by two-way airlines. Suppose that for each k with $2 \le k \le N$ and any k cities there are at most $2k - 2$ airlines mutually connecting these cities. Prove that all airlines can be assigned to two companies so that no company can serve a round route by itself.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com