27-th All-Russian Mathematical Olympiad 2001

Final Round – Tver, April 21–22

Grade 9

First Day

1. The integers from 1 to 999999 are partioned into two groups: the first group consists of those integers for which the closest perfect square is odd, whereas the second group consists of those for which the closest perfect square is even. In which group is the sum of the elements greater? *(N. Agakhanov)*

2. The two polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ take negative values on an interval I of length greater than 2, and nonnegative values outside I. Prove that there exists $x_0 \in \mathbb{R}$ such that $P(x_0) < Q(x_0)$. *(N. Agakhanov)*

3. A point K is taken inside a parallelogram $ABCD$ so that the midpoint of AD is equidistant from K and C, and the midpoint of CD is equidistant from K and A. Let N be the midpoint of BK. Prove that the angles NAK and NCK are equal. *(S. Berlov)*

4. Let be given a convex 2000-gon, no three of whose diagonals have a common point. Each of its diagonals is colored in one of 999 colors. Prove that there exists a triangle whose all sides lie on diagonals of the same color. (Vertices of the triangle need not be vertices of the initial polygon.) *(Y. Lifshits)*

Second Day

5. Yura put 2001 coins of 1, 2 or 3 kopeykas in a row. It turned out that between any two 1-kopeyka coins there is at least one coin; between any two 2-kopeykas coins there are at least two coins; and between any two 3-kopeykas coins there are at least 3 coins. How many 3-kopeykas coins could Yura put? *(Y. Lifshits)*

6. In a set of $2n+1$ persons, for any n persons there exists a person different from them who knows each of them. Prove that there is a person in this set who knows all the others. *(S. Berlov)*

7. Let N be a point on the longest side AC of a triangle ABC. The perpendicular bisectors of AN and NC intersect AB and BC respectively in K and M. Prove that the circumcenter O of $\triangle ABC$ lies on the circumcircle of triangle KBM. *(S. Berlov)*

8. Find all odd integers $n > 1$ such that, whenever a and b are coprime divisors of n, the number $a+b-1$ is also a divisor of n. *(D. Djukić)*

Grade 10

First Day

1. *Problem 1 for Grade 9.*

2. Let 100 sets $A_1, A_2, \dots, A_{100}$ be given on a line. Each of the sets is a union of 100 pairwise disjoint closed segments. Prove that the intersection of $A_1, A_2, \dots, A_{100}$ is a union of at most 9901 pairwise disjoint segments. (A point is also considered a segment.) *(R. Karasev)*

3. Two circles are internally tangent to each other at N. The tangent to the internal circle at point K cuts the external circle in A and B. Let M be the midpoint of the arc AB not containing N. Prove that the circumradius of $\triangle BMK$ does not depend on the choice of K on the internal circle. *(T. Emelyanova)*

4. Some towns in a country are connected by two-way roads, so that for any two towns there is a unique path along the roads connecting them. It is known that there are exactly 100 towns which are directly connected to only one town. Prove that we can construct 50 new roads in order to obtain a net in which every two towns will be connected even if one road gets closed. *(D. Karpov)*

Second Day

5. A polynomial $P(x) = x^3 + ax^2 + bx + c$ has three distinct real roots, while the polynomial $P(Q(x))$ has no real roots, where $Q(x) = x^2 + x + 2001$. Show that $P(2001) > 1/64$. *(D. Tereshin)*

6. In a magic square $n \times n$ composed from the numbers $1, 2, \dots, n^2$, the centers of any two squares are joined by a vector going from the smaller number to the bigger one. Prove that the sum of all these vectors is zero. (A magic square is a square matrix such that the sums of entries in all its rows and columns are equal.) *(S. Berlov)*

7. Points A_1, B_1, C_1 inside an acute-angled triangle ABC are selected on the altitudes from A, B, C respectively so that the sum of the areas of triangles ABC_1, BCA_1, and CAB_1 is equal to the area of triangle ABC. Prove that the circumcircle of triangle $A_1B_1C_1$ passes through the orthocenter H of $\triangle ABC$. *(S. Berlov)*

8. Find all natural numbers n such that, whenever a and b are coprime divisors of n, the number $a + b - 1$ is also a divisor of n. *(D. Djukić)*

Grade 11

First Day

1. The total mass of 100 given weights with positive masses equals $2S$. A natural number k is called *middle* if some k of the given weights have the total mass S. Find the maximum possible number of middle numbers.

(D. Kuznetsov)

2. *Problem 3 for Grade 10.*

3. Two families $\mathscr{P}_1$ and $\mathscr{P}_2$ of convex polygons are given on the plane. Any two polygons from different families have a nonempty intersection. Moreover, in each of the two families there exist two disjoint polygons. Prove that there exists a line which cuts all the polygons in both families.

(V. Dolnikov)

4. Participants of an olympiad worked on n problems. Each problem was worth an integer number of points determined by the jury. A contestant gets 0 points for a wrong answer and all points for a correct answer to a problem. It turned out after the olympiad that the jury could impose the worths of the problems so as to obtain any (strict) final ranking of the contestants. Find the greatest possible number of the contestants.

(S. Tokarev)

Second Day

5. Two monic quadric trinomials $f(x)$ and $g(x)$ take negative values on disjoint intervals. Prove that there exist positive numbers α and β such that $\alpha f(x) + \beta g(x) > 0$ for all real x. *(S. Berlov, O.Podlipskiy)*

6. Let a and b be two distinct natural numbers such that $ab(a+b)$ is divisible by a^2+ab+b^2. Prove that $|a-b| > \sqrt[3]{ab}$. *(S. Berlov)*

7. The 2001 towns in a country are connected by some roads, at least one road from each town, so that no town is connected by a road to every other city. We call a set D of towns *dominant* if every town not in D is connected by a road to a town in D. Suppose that each dominant set consists of at least k towns. Prove that the country can be partitioned into $2001-k$ republics in such a way that no two towns in the same republic are connected by a road. *(V. Dolnikov)*

8. A sphere with center on the plane of the face ABC of a tetrahedron $SABC$ passes through A, B and C, and meets the edges SA, SB, SC again at A_1, B_1, C_1, respectively. The planes through A_1, B_1, C_1 tangent to the sphere meet at a point O. Prove that O is the circumcenter of the tetrahedron $SA_1B_1C_1$. *(L. Emelyanov)*

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