Romanian IMO Team Selection Tests 1999

First Test

Time: 4 hours

1. (a) Show that among any 39 consecutive natural numbers there exists one having the sum of digits divisible by 11.

   (b) Find the first 38 consecutive natural numbers none of which has the sum of digits divisible by 11.

2. In an acute-angled triangle ABC, the bisectors of interior angles at B and C meet the opposite sides at L and M respectively. Prove that there is a point $K \in BC$ such that the triangle KLM is equilateral if and only if $\angle A = 60^\circ$.

3. Prove that for any positive integer n, the number

$$S_n = \binom{2n+1}{0} 2^{2n} + \binom{2n+1}{2} 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} 3^n$$

   is the sum of two consecutive squares.

4. If $x_1, x_2, \ldots, x_n$ are positive real numbers with the product 1, show that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

Second Test

Time: 4 hours

1. Show that for any distinct positive integers $x_1, x_2, \ldots, x_n$ it holds that

$$\sum_{k=1}^{n} x_k^2 \geq \frac{2n+1}{3} \sum_{k=1}^{n} x_k.$$

2. In a triangle ABC, H is the orthocenter, O is the circumcenter and R is the circumradius. Points D, E, F are reflections of A, B, C across the opposite sides, respectively. Show that D, E and F are collinear if and only if $OH = 2R$.

3. Prove that for any $n \geq 3$ there exist an arithmetic progression of n positive integers $a_1, a_2, \ldots, a_n$ and a geometric progression of n positive integers $b_1, b_2, \ldots, b_n$ such that $b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n$. Give an example of two such progressions for $n = 5$.

Third Test

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Time: 4 hours

1. For $a > 0$, let a sequence (x_n) be such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k \quad \text{for all } n \geq 1.$$

Show that there exists a positive integer n for which $x_n > 1999!$.

2. Let O, A, B, C be variable points in the plane such that $OA = 4$, $OB = 2\sqrt{3}$ and $OC = \sqrt{22}$. Find the maximum area of triangle ABC.

3. Determine all positive integers n for which there exists an integer a such that

$$2^n - 1 \mid a^2 + 9.$$

Fourth Test

Time: 4 hours

1. Let a and n be natural numbers and p be prime such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ is irreducible over $\mathbb{Z}[x]$.

2. Two circles intersect at points A, B. A line passing through A meets the circles again at C and D. Let M and N be the midpoints of arcs BC and BD not containing A, and K be the midpoint of CD. Show that $\angle MKN = 90^\circ$.

3. Let $A_1, A_2, \ldots, A_n$ be points on a circle ($n \geq 3$). Find the greatest possible number of acute-angled triangles with vertices in these points.

Fifth Test

Time: 4 hours

1. The participants of an international conference are native or foreign. Each native scientist sends a message to a foreign one, and vice-versa. There are native scientists who did not receive any message. Prove that there exists a set S of native scientists such that the scientists not in S are exactly those who received messages from those foreign scientists who received messages from scientists in S.

2. Let X be a set of n elements and $A_1, A_2, \ldots, A_m$ be three-element subsets of X, any two of which have at most one element in common. Prove that there exists a subset A of X with at least $\left[\sqrt{2n}\right]$ elements which does not contain any of the A_i's.

3. Let P be an arbitrary convex polyhedron in the space. Decide whether there always exist three edges of P which are sides of a triangle.

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