Romanian IMO Team Selection Tests 1998

First Test

Time: 4 hours

1. Consider all words of length n over the alphabet $\{a,b,c\}$. Let us denote by A_n the set of such words which do not contain any block of type aa or bb and by B_n the set of words in which no three consecutive letters are mutually distinct. Prove that $|B_{n+1}| = 3|A_n|$.

2. Both the volume and area of a parallelepiped are equal to 216. Prove that this parallelepiped is a cube.

3. Given an integer $m \geq 2$, find the smallest integer $n > m$ such that for any partition of the set $\{m, m+1, \dots, n\}$ into two classes at least one of the classes contains three numbers a,b,c (not necessarily distinct) such that $a^b = c$.

4. Let be given a finite set of segments in the plane such that the sum of their lengths is less than $\sqrt{2}$. Show that there is an infinite unit square grid covering the plane so that the grid lines do not intersect any of the given segments.

Second Test

Time: 4 hours

1. Let ABC be an isosceles triangle with $BC = a$, $AB = AC = b$. Find the locus of the intersection P ofd the lines BM and CN, where M,N are points on sides AC,BC respectively such that $a^2 AN \cdot AM = b^2 BN \cdot CM$.

2. The vertices of a convex pentagon have integer coordinates. Prove that the area of the pentagon is at least $5/2$.

3. Find all pairs (x,n) of positive integers satisfying $x^n + 2^n + 1 \mid x^{n+1} + 2^{n+1} + 1$.

Third Test

Time: 4 hours

1. Let $n \geq 2$ be an integer. Show that there exists a subset A of $\{1,2,\dots,n\}$ such that:

   (i) A has at most $2\left[\sqrt{n}\right] + 1$ elements;

   (ii) the set of differences $|x-y|$, where $x,y \in A$, is exactly $\{0,1,\dots,n-1\}$.

2. An infinite arithmetic progression of positive integers contains a perfect square and a perfect cube. Prove that it also contains the sixth power of an integer.

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3. Show that for any $n \in \mathbb{N}$ the polynomial $f(x) = (x^2 + x)^{2^n} + 1$ is irreducible over $\mathbb{Z}[x]$.

Fourth Test

Time: 4 hours

1. Let ABC be an equilateral triangle. For an integer $n \geq 2$, consider the set $\mathscr{A}$ of $n-1$ lines parallel to BC which divide the triangle into n parts of equal areas, and the set $\mathscr{P}$ the set of $n-1$ lines parallel to BC which divide the triangle into n parts of equal parameters. Prove that $\mathscr{A}$ and $\mathscr{P}$ are disjoint.

2. Let $a_1 < a_2 < \cdots < a_n$ be integers, where $n \geq 3$. Prove that these integers form an arithmetic progression if and only if there exists a partition of $\mathbb{N}$ into sets $A_1, A_2, \ldots, A_n$ such that
$$a_1 + A_1 = a_2 + A_2 = \cdots = a_n + A_n.$$
(As usual, for a set X and a number a we define $a + X = \{a + x \mid x \in X\}$.)

3. For a positive integer n, let $\mathscr{P}_n$ be the set of polynomials $a_0 + a_1 + \cdots + a_n x^n$ such that $|a_i| \leq 2$ for $i = 0, 1, \ldots, n$. For each $k \in \mathbb{N}$, find the number of elements of the set $\{f(k) \mid f \in \mathscr{P}_n\}$.

Fifth Test

Time: 4 hours

1. Find all functions $u : \mathbb{R} \to \mathbb{R}$ for shich there exists a monotonous function $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x+y) = f(x)u(y) + f(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

2. Find all positive integers k with the following condition: If f is a polynomial with integer coefficients such gthat $0 \leq f(a) \leq k$ for each $a = 0, 1, \ldots, k+1$, then $f(0) = f(1) = \cdots = f(k+1)$.

3. The lateral survace of a cylinder is divided by $n-1$ planes parallel to the base and m meridians into mn cells ($n \geq 1$, $m \geq 3$). Two cells are called neighbors if they have a common side. Prove that it is possible to write real numbers in the cells, not all zero, so that the number in each cell equals the sum of the numbers in the neighboring cells, if and only if there exist k, l with $n+1 \nmid k$ such that $\cos \dfrac{2l\pi}{m} + \cos \dfrac{k\pi}{n+1} = \dfrac{1}{2}$.

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