Romanian IMO Team Selection Tests 1997

First Test

Time: 4.5 hours

1. In the plane are given a line Δ and three circles tangent to Δ and externally tangent to each other. Show that the triangle whose vertices are the centers of the circles has an obtuse angle, and find the maximum value of that angle.

2. Find all sets A of 9 positive integers such that for any positive integer $n \leq 500$ there is a subset of A whose elements sum up to n.

3. Let M be a set of n points in the plane, $n \geq 4$, no three of which are collinear and not all lying on a circle. Suppose that $f : M \to \mathbb{R}$ is a function such that for any circle C passing through at least three points of M,
$$\sum_{P \in M \cap C} f(P) = 0.$$
Show that $f \equiv 0$.

4. Let D be a point on the side BC of a triangle ABC and let ω be the circumcircle of $\triangle ABC$. Let γ_1 be the circle tangent to AD, BD and internally to ω, and γ_2 be the circle tangent to AC, CD and internally to ω. Show that γ_1 and γ_2 are tangent if and only if AD bisects the angle BAC.

Second Test

Time: 4.5 hours

1. Let be given a pyramid $VA_1 \dots A_n$, $n \geq 4$. A plane Π intersects the edges $VA_1, \dots, VA_n$ at points $B_1, \dots, B_n$, respectively. Suppose that the polygons $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$ are similar. Prove that Π is parallel to the plane $A_1A_2 \dots A_n$.

2. Let A be the set of integers that can be written as $a^2 + 2b^2$ for some integers a, b, where $b \neq 0$. Show that if p is a prime and $p^2 \in A$ then $p \in A$.

3. Let $p \geq 5$ be a prime and k be an integer with $0 \leq k < p$. Find the maximum length of an arithmetic progression, none of whose terms contain a digit k in base p.

4. Let p, q, r be distinct prime numbers. Consider the set
$$A = \{p^a q^b r^c \mid 0 \leq a, b, c \leq 5\}.$$
Find the smallest $n \in \mathbb{N}$ such that any n-element subset of A contains two distinct elements x, y such that x divides y.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

Third Test

Time: 4.5 hours

1. Let $ABCDEF$ be a convex hexagon. Let P,Q,R be the intersection points of AB and EF, EF and CD, CD and AB, respectively, and let S,T,U be the intersection points of BC and DE, DE and FA, FA and BC, respectively. Show that if $\frac{AB}{PR} = \frac{CD}{RQ} = \frac{EF}{QP}$, then $\frac{BC}{US} = \frac{DE}{ST} = \frac{FA}{TU}$.

2. Let P and D denote the set of all points and the set of all lines in the plane, respectively. Does there exist a bijective function $f : P \to D$ such that, whenever A,B,C are collinear points, lines $f(A), f(B), f(C)$ are either concurrent or parallel?

3. Find all functions $f : \mathbb{R} \to [0,\infty)$ such that
$$f(x^2+y^2) = f(x^2-y^2) + f(2xy) \qquad \text{for all } x,y \in \mathbb{R}.$$

4. Let $n \geq 2$ be an integer and $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1$ be a polynomial with integer coefficients. Suppose that $a_k = a_{n-k}$ for $k = 1,\ldots,n-1$. Prove that there exist infinitely many pairs of positive numbers (x,y) satisfying $x \mid P(y)$ and $y \mid P(x)$.

Fourth Test

Time: 4.5 hours

1. Let $P(x)$ and $Q(x)$ be monic irreducible polynomials with rational coefficients. Suppose that there are roots α of P and β of Q such that $\alpha+\beta$ is rational. Prove that $P(x)^2 - Q(x)^2$ has a rational root.

2. Let $a > 1$ be an integer. Prove that the set $\{a^{n+1} + a^n - 1 \mid n \in \mathbb{N}\}$ contains an infinite subset of pairwise coprime numbers.

3. Determine the number of ways to color the vertices of a regular 12-gon in two colors so that no set of vertices of the same color form a regular polygon.

4. Let Γ be a circle and AB a line not meeting Γ. For any point P on Γ, let the line AP meets Γ again at P' and let the line BP' meet Γ again at $f(P)$. Given a point $P_0 \in \Gamma$, we define the sequence P_i by $P_{n+1} = f(P_n)$ for $n \in \mathbb{N}_0$. Show that if $k > 0$ is an integer such that $P_k = P_0$ holds for a single choice of P_0, then $P_k = P_0$ holds for every choice of P_0.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com