Romanian IMO Team Selection Tests 1996

First Test

Time: 4.5 hours

1. Let $n > 2$ be an integer and $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon $A_1A_2\dots A_n$,
$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$
Prove that f is the zero function.

2. Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \dots, x_n$, not all zero, such that for any $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ from the set $\{-1, 0, 1\}$, not all zero, $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$ is not divisible by n^3.

3. Prove that if the set $\{\cos(n\pi x) + \cos(n\pi y) \mid n \in \mathbb{N}\}$ is finite for some real numbers x, y, then x, y are rational.

4. Let $ABCD$ be a cyclic quadrilateral and let $\mathscr{M}$ be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (16 points in total). Prove that there are two sets $\mathscr{K}$ and $\mathscr{L}$ of four parallel lines each, such that every line in $\mathscr{K} \cup \mathscr{L}$ contains exactly four points of $\mathscr{M}$.

Second Test

Time: 4.5 hours

1. Let A and B be points on a circle $\mathscr{C}$ with center O such that $\angle AOB = \pi/2$. Circles $\mathscr{C}_1$ and $\mathscr{C}_2$ are internally tangent to $\mathscr{C}$ at A and B respectively, and a circle $\mathscr{C}_3$ is externally tangent to $\mathscr{C}_1, \mathscr{C}_2$ at S and T and internally tangent to $\mathscr{C}$ at M. Determine $\angle SMT$.

2. Let $\mathscr{C}$ be a circle with center O. A line d intersects the circle $\mathscr{C}$ at C and D and the diameter AB of $\mathscr{C}$ at M so that $MB < MA$ and $MD < MC$. The circumcircles of AOC and BOD intersect again at K. Prove that OK is perpendicular to KM.

3. Let a be a real number and $f_1, f_2, \dots, f_n : \mathbb{R} \to \mathbb{R}$ be additive functions such that
$$f_1(x) f_2(x) \cdots f_n(x) = ax^n \quad \text{for all } x \in \mathbb{R}.$$
Prove that there exist $i \in \{1, 2, \dots, n\}$ and $b \in \mathbb{R}$ such that $f_i(x) = bx$ for all real x.

4. The sequence $(a_n)_{n\ge 2}$ is defined as $a_n = p_1^{-1} + p_2^{-1} + \cdots + p_k^{-1}$, where $p_1, p_2, \dots, p_k$ are the distinct prime factors of n. Show that for any integer $N \ge 2$,
$$a_2 + a_2a_3 + \cdots + a_2a_3\cdots a_N < 1.$$

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Third Test

Time: 4.5 hours

1. Let $x_1, x_2, \dots, x_{n-1}$ ($n \geq 3$) be nonnegative integers such that
$$\begin{aligned} x_1 + x_2 + \cdots + x_{n-1} &= n, \\ x_1 + 2x_2 + \cdots + (n-1)x_{n-1} &= 2n-2. \end{aligned}$$
Find the minimum value of $F(x_1, \dots, x_{n-1}) = \sum_{k=1}^{n-1} k(2n-k)x_k$.

2. Let n and r be positive integers and A be a set of lattice points in the plane such that any open disc of radius r contains a point of A. Show that for any coloring of the points of A in n colors there exists four points of the same color which are the vertices of a rectangle.

3. Find all primes p, q such that $3pq \mid \alpha^{3pq} - \alpha$ for all integers α.

4. Let $n \geq 3$ be an integer and $p \geq 2n-3$ be a prime. For a set M of n points in the plane, no three collinear, let $f : M \to \{0, 1, \dots, p-1\}$ be a function such that

   (i) exactly one point of M maps to zero, and

   (ii) if a circle k passes through distinct points $A, B, C \in M$, then $\sum_{P \in M \cap k} f(P) \equiv 0 \pmod{p}$.

   Show that all the points of M lie on a circle.

Fourth Test

Time: 4.5 hours

1. Let $x_1, x_2, \dots, x_n$ be positive real numbers and $x_{n+1} = x_1 + x_2 + \cdots + x_n$. Prove that
$$\sum_{i=1}^{n} \sqrt{x_i(x_{n+1} - x_i)} \leq \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1} - x_i)}\,.$$

2. Let x, y, z be real numbers. Prove that the following conditions are equivalent:

   (i) x, y, z are positive and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 1$.

   (ii) $a^2x + b^2y + c^2z > d^2$ holds for every quadrilateral with sides a, b, c, d.

3. Let $\mathscr{D}$ be a set of n concentric circles in the plane. Prove that if the function $f : \mathscr{D} \to \mathscr{D}$ satisfies
$$d\left(f(A), f(B)\right) \geq d(A, B) \quad \text{for all } A, B \in \mathscr{D},$$
then $d\left(f(A), f(B)\right) = d(A, B)$ for all $A, B \in \mathscr{D}$.

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4. Let $n \geq 3$ be an integer and X be a $3n^2$-element subset of $\{1,2,\dots,n^3\}$. Prove that there are nine distinct numbers $a_1,\dots,a_9 \in X$ such that the system

$$\begin{aligned} a_1x+a_2y+a_3z &= 0 \\ a_4x+a_5y+a_6z &= 0 \\ a_7x+a_8y+a_9z &= 0 \end{aligned}$$

has a solution (x_0,y_0,z_0) in nonzero integers.

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