Romanian IMO Team Selection Tests 1995

First Test
Bacau, March 31

1. Let AD be the altitude of a triangle ABC and E, F be the incenters of the triangles ABD and ACD, respectively. Line EF meets AB and AC at K and L. Prove that $AK = AL$ if and only if $AB = AC$ or $\angle A = 90°$.

2. Find all positive integers x, y, z, t such that x, y, z are pairwise coprime and
$$(x + y)(y + z)(z + x) = txyz.$$

3. Let $n \geq 6$ and $3 \geq p < n - p$ be two integers. The vertices of a regular n-gon are colored so that p vertices are red and the others are black. Prove that there exist two congruent polygons with at least $[p/2] + 1$ vertices, one with all the vertices red and the other with all the vertices black.

4. Let m, n be positive integers greater than 2. Find the number of polynomials of degree $2n-1$ with distinct coefficients from the set $\{1, 2, \dots, m\}$, which are divisible by $x^{n-1} + x^{n-2} + \cdots + 1$.

Second Test
Bucharest, April 21

1. The sequence (x_n) is defined by $x_1 = 1$, $x_2 = a$ and
$$x_n = (2n + 1)x_{n-1} - (n^2 - 1)x_{n-2}, \qquad n \geq 3,$$
where a is a positive integer. For which values of a does the sequence have the property that $x_i \mid x_j$ whenever $i < j$?

2. Suppose that n polygons of area $s = (n - 1)^2$ are placed on a polygon of area $S = \dfrac{n(n-1)^2}{2}$. Prove that there exist two of the n smaller polygons whose intersection has the area $\sigma \geq 1$.

3. Let M, N, P, Q be points on sides AB, BC, CD, DA of a convex quadrilateral $ABCD$ such that $AQ = DP = CN = BM$. Prove that if $MNPQ$ is a square, then $ABCD$ is also a square.

4. A convex set S on a plane, not lying on a line, is painted in p colors. Prove that for every $n \geq 3$ there exist infinitely many congruent n-gons whose vertices are of the same color.

Third Test
Bucharest, May 18

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imo.org.yu

1. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers. Prove that

$$(a_1^5 + \cdots + a_n^5) + (a_1^7 + \cdots + a_n^7) \geq 2(a_1^3 + \cdots + a_n^3)^2,$$

and find the cases of equality.

2. A cube is partitioned into finitely many rectangular parallelepipeds with the edges parallel to the edges of the cube. Prove that if the sum of the volumes of the circumspheres of these parallelepipeds equals the volume of the circumscribed sphere of the cube, then all the parallelepipeds are cubes.

3. Let f be an irreducible (in $\mathbb{Z}[x]$) monic polynomial with integer coefficients and of odd degree greater than 1. Suppose that the modules of the roots of f are greater than 1 and that $f(0)$ is a square-free number. Prove that the polynomial $g(x) = f(x^3)$ is also irreducible.

4. Find a sequence of positive integers $f(n)$ $(n \in \mathbb{N})$ such that:

   (i) $f(n) \leq n^8$ for any $n \geq 2$;

   (ii) for any distinct $a_1, \ldots, a_k, n$, $f(n) \neq f(a_1) + \cdots + f(a_k)$.

Fourth Test
Bucharest, May 19

1. How many colorings of an n-gon in $p \geq 2$ colors are there such that no two neighboring vertices have the same color?

2. For each positive integer n, define $f(n) = \text{lcm}\,[1, 2, \ldots, n]$.

   (a) Prove that for every k there exist k consecutive positive integers on which f is constant.

   (b) Find the maximum possible cardinality of a set of consecutive positive integers on which f is strictly increasing and find all sets for which this maximum is attained.

3. The altitudes of a triangle have integer length and its inradius is a prime number. Find all possible values of the sides of the triangle.

4. Let $ABCD$ be a convex quadrilateral. Suppose that similar isosceles triangles APB, BQC, CRD, DSA with the bases on the sides of $ABCD$ are constructed in the exterior of the quadrilateral such that $PQRS$ is a rectangle but not a square. Show that $ABCD$ is a rhombus.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imo.org.yu