Romanian Team Selection Tests 1994

Selection Test for Balkan MO
Arad, March 31

1. Prove that if n is a square-free positive integer, there are no coprime positive integers x and y such that $(x+y)^3$ divides x^n+y^n.

2. Let $n \geq 4$ be an integer. Find the maximum possible area of an n-gon inscribed in a unit cicle and having two perpendicular diagonals.

3. Let $M_1, M_2, \dots, M_{11}$ be 5-element sets such that $M_i \cap M_j \neq \emptyset$ for all $i, j \in \{1, \dots, 11\}$. Determine the minimum possible value of the greatest number of the given sets that have nonempty intersection.

4. Consider a tetrahedron $A_1A_2A_3A_4$. A point N is said to be a *Servais point* if its projections on the six edges of the tetrahedron lie in a plane $\alpha(N)$ (called *Servais plane*). Prove that if all the six points N_{ij} symmetric to a point M with respect to the midpoints B_{ij} of the edges A_iA_j are Servais points, then M is contained in all Servais planes $\alpha(N_{ij})$.

First Test for IMO
Bucharest, May 28

1. Let $X_n = \{1, 2, \dots, n\}$, where $n \geq 3$. We define the measure $m(X)$ of $X \subset X_n$ as the sum of its elements, where $m(\emptyset) = 0$. A set $X \subset X_n$ is said to be *even* (resp. *odd*) if $m(X)$ is even (resp. odd).
   (a) Show that the number of even sets equals the number of odd sets.
   (b) Show that the sum of the measures of the even sets equals the sum of the measures of the odd sets.
   (c) Compute the sum of the measures of the odd sets.

2. Let n be a positive odd number. Show that the number $n(n-1)^{(n-1)^n+1}+n$ is divisible by $((n-1)^n+1)^2$. Is this necessarily true if n is even?

3. Prove that the sequence $a_n = 3^n - 2^n$ contains no three numbers in geometric progression.

4. Inscribe an equilateral triangle of minimum side in a given acute-angled triangle ABC (one vertex on each side).

Second Test for IMO
Bucharest, May 29

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1. Find the minimal term in the sequence defined by $a_1 = 1993^{1994^{1995}}$ and

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even,} \\ a_n + 7 & \text{if } a_n \text{ is odd.} \end{cases}$$

2. Let S_1, S_2, S_3 be spheres of radii a, b, c respectively whose centers lie on a line l. Sphere S_2 is externally tangent to S_1 and S_3, whereas S_1 and S_3 have no common points. A straight line t touches each of the spheres, Find the sine of the angle between l and t.

3. Let $a_1, a_2, \ldots, a_n$ be a finite sequence of 0 and 1. Under any two consecutive terms of this sequence 0 is written if the digits are equal and 1 is written otherwise. This way a new sequence of length $n-1$ is obtained. By repeating this procedure $n-1$ times one obtains a triangular table of 0 and 1. Find the maximum possible number of ones that can appear on this table.

4. Let be given two concentric circles of radii R and $R_1 > R$. Let quadrilateral $ABCD$ is inscribed in the smaller circle and let the rays CD, DA, AB, BC meet the larger circle at A_1, B_1, C_1, D_1, respectively. Prove that
$$\frac{\sigma(A_1B_1C_1D_1)}{\sigma(ABCD)} \geq \frac{R_1^2}{R^2},$$
where $\sigma(P)$ denotes the area of a polygon P.

Third Test for IMO
Bucharest, May 30

1. Let p be a (positive) prime number. Suppose that real numbers $a_1, a_2, \ldots, a_{p+1}$ have the property that, whenever one of the numbers is deleted, the remaining numbers can be partitioned into two classes with the same arithmetic mean. Show that these numbers must be equal.

2. Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ for which $P(2) = n$.

3. Determine all integer solutions of the equation $x^n + y^n = 1994$, where $n \geq 2$.

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