Romanian Team Selection Tests 1993

Selection Test for Balkan MO
Constanţa

1. Consider the sequence $z_n = (1+i)(2+i)\cdots(n+i)$. Prove that the sequence $\operatorname{Im} z_n$ contains infinitely many positive and infinitely many negative numbers.
2. The circumcircle and incircle of a triangle ABC have radii R and r respectively. Denote $d = \frac{rR}{r+R}$. Prove that there exists a triangle DEF such that for any point M inside $\triangle ABC$ there is a point X on the sides of DEF such that $MX \leq d$.
3. Show that the set $\{1, 2, \dots, 2^n\}$ can be partitioned in two classes, none of which contains an arithmetic progression of length $2n$.
4. Prove that the equation $(x+y)^n = x^m + y^m$ has a unique solution in integers with $x > y > 0$ and $m, n > 1$.

First Test for IMO
Bucureşti, June 1

1. Find the greatest number $A > 0$ such that for all positive real numbers x, y, z,
$$\frac{x}{\sqrt{y^2+z^2}} + \frac{y}{\sqrt{z^2+x^2}} + \frac{z}{\sqrt{x^2+y^2}} > A.$$
2. If x, y, z are integers such that $x^2 + y^2 + z^2 = 1993$, prove that $x + y + z$ is not a perfect square.
3. Suppose that each of the diagonals AD, BE, CF divides the hexagon $ABCDEF$ into two parts of the same area and perimeter. Does the hexagon necessarily have a center of symmetry?
4. Let $\mathscr{Y}$ be the family of all subsets of $X = \{1, 2, \dots, n\}$ $(n > 1)$ and let $f : \mathscr{Y} \to X$ be an arbitrary mapping. Prove that there exist distinct subsets A, B of X such that $f(A) = f(B) = \max A \Delta B$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Second Test for IMO
Bucureşti, June 2

1. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a strictly increasing function such that $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$ for all $x, y > 0$. Prove that the sequence $a_n = f(n)$ $(n \in \mathbb{N})$ does not contain an infinite arithmetic progression.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

2. For coprime integers $m > n > 1$ consider the polynomials

$$f(x) = x^{m+n} - x^{m+1} - x + 1 \quad \text{and} \quad g(x) = x^{m+n} + x^{n+1} - x + 1.$$

If f and g have a common divisor of degree greater than 1, find this divisor.

3. Find all integers $n > 1$ for which there is a set $\mathscr{B}$ of n points in the plane such that for any $A \in \mathscr{B}$ there are three points $X, Y, Z \in \mathscr{B}$ with $AX = AY = AZ = 1$.

4. For each integer $n > 3$ find all quadruples (n_1, n_2, n_3, n_4) of positive integers with $n_1 + n_2 + n_3 + n_4 = n$ which maximize the expression

$$\frac{n!}{n_1!n_2!n_3!n_4!} 2^{\binom{n_1}{2}+\binom{n_2}{2}+\binom{n_3}{2}+\binom{n_4}{2}+n_1n_2+n_2n_3+n_3n_4}.$$

Third Test for IMO

Bucureşti, June 3

1. Define the sequence (x_n) as follows: the first term is 1, the next two are $2, 4$, the next three are $5, 7, 9$, the next four are $10, 12, 14, 16$, and so on. Express x_n as a function of n.

2. Suppose that D, E, F are points on sides BC, CA, AB of a triangle ABC respectively such that $BD = CE = AF$ and $\angle BAD = \angle CBE = \angle ACF$. Prove that the triangle ABC is equilateral.

3. Let $p \geq 5$ be a prime number. Prove that for any partition of the set $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$ in three subsets there exist numbers x, y, z, each belonging to a distinct subset, such that $x + y \equiv z \pmod{p}$.