Romanian Team Selection Tests 1992

Selection Test for Balkan MO

1. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is an increasing function such that $f(f(n)) = 3n$ for all n. Find $f(1992)$.

2. For a positive integer a, define the sequence (x_n) by $x_1 = x_2 = 1$ and
$$x_{n+2} = (a^4 + 4a^2 + 2)x_{n+1} - x_n - 2a^2, \qquad \text{for } n \geq 1.$$
Show that x_n is a perfect square and that for $n > 2$ its square root equals the first entry in the matrix $\begin{pmatrix} a^2+1 & a \\ a & 1 \end{pmatrix}^{n-2}$.

3. In a tetrahedron $ABCD$, let B', C', D' be the midpoints of edges AB, AC, AD and let G_a, G_b, G_c, G_d, G be the barycenters of $BCD, ACD, ABD, ABC, ABCD$, respectively. Show that A, G, G_b, G_c, G_d lie on a sphere if and only if A, G, B', C', D' lie on a sphere.

4. Let $x_1, x_2, \dots, x_n$ be real numbers with $1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. If $[x_1 + x_2 + \cdots + x_n] = m$, prove that
$$x_1 + x_2 + \cdots + x_m \geq 1.$$

First Test for IMO

1. Let $S > 1$ be a real number. The Cartesian plane is partitioned into rectangles whose sides are parallel to the axes of the coordinate system. and whose vertices have integer coordinates. Prove that if the area of each triangle if at most S, then for any positive integer k there exist k vertices of these rectangles which lie on a line.

2. Suppose $a_1, a_2, \dots, a_n$ are distinct positive integers such that the 2^n sums $\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n$ ($\varepsilon_i \in \{0, 1\}$ for each i) are pairwise distinct. Prove that
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq 2 - \frac{1}{2^{n-1}},$$
and find when equality holds.

3. Let π be the set of points in a plane and $f : \pi \to \pi$ be a mapping such that the image of any triangle (as its polygonal line) is a square. Show that $f(\pi)$ is a square.

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4. Let A be the set of all ordered sequences $(a_1, a_2, \dots, a_{11})$ of zeros and ones. The elements of A are ordered as follows: The first element is $(0,0,\dots,0)$, and the $n+1-$th is obtained from the $n-$th by changing the first component from the right such that the newly obtained sequence was not obtained before. Find the $1992-$th term of the ordered set A.

Second Test for IMO

1. Let O be the circumcenter of an acute triangle ABC. Suppose that the circumradius of the triangle is $R = 2p$, where p is a prime number. The ines AO, BO, cO meet the sides BC, CA, AB at A_1, B_1, C_1, respectively. Given that the lengths of OA_1, OB_1, OC_1 are positive integers, find the side lengths of the triangle.

2. Let m,n be positive integers and p be a prime number. Show that if $\dfrac{7^m + p\cdot 2^n}{7^m - p\cdot 2^n}$ is an integer, then it is a prime number.

3. Let the sequences (a_n) and (b_n) of positive integers satisfy

$$a_{n+1} = na_n + 1 \quad \text{and} \quad b_{n+1} = nb_n - 1.$$

Show that these two sequences have only finitely many terms in common.

4. Let $m,n \geq 2$ be integers. The sides $A_{00}A_{0m}$ and $A_{nm}A_{n0}$ of a convex quadrilateral $A_{00}A_{0m}A_{nm}A_{n0}$ are divided into m equal segments by points A_{0j} and A_{nj} respectively ($j = 1, \dots, m-1$). The other two sides are divided into n equal segments by points A_{i0} and A_{im} ($i = 1, \dots, n-1$). Denote by A_{ij} the intersection of lines $A_{0j}A_{nj}$ and $A_{i0}A_{im}$, by S_{ij} the area of quadrilateral $A_{ij}A_{i,j+1}A_{i+1,j+1}A_{i+1,j}$ and by S the area of the big quadrilateral. Show that

$$S_{ij} + S_{n-1-i,m-1-j} = \frac{2S}{mn}.$$

Third Test for IMO

1. Let x,y be real numbers such that $1 \leq x^2 - xy + y^2 \leq 2$. Prove that

   (a) $\dfrac{2}{9} \leq x^4 + y^4 \leq 8$;

   (b) $x^{2n} + y^{2n} \geq \dfrac{2}{3^n}$ for all $n \geq 3$.

2. In a tetrahedron $VABC$, let I be the incenter and A', B', C' be arbitrary points on the edges AV, BV, CV, and let S_a, S_b, S_c, S_v be the areas of triangles VBC, VAC, VAB, ABC, respectively. Show that points A', B', C', I are coplanar if and only if

$$\frac{AA'}{A'V}S_a + \frac{BB'}{B'V}S_b + \frac{CC'}{C'V}S_c = S_v.$$

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3. In the Cartesian plane is given a polygon $\mathscr{P}$ whose vertices have integer coordinates and with sides parallel to the coordinate axes. Show that if the length of each edge of $\mathscr{P}$ is an odd integer, then the surface of $\mathscr{P}$ cannot be partitioned into 2×1 rectangles.

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