Romanian Team Selection Tests 1991

Selection Test for Balkan MO

1. Suppose that a,b are positive integers for which $A = \dfrac{a+1}{b} + \dfrac{b}{a}$ is an integer. Prove that $A = 3$.

2. Let $A_1A_2A_3A_4$ be a tetrahedron. For any permutation (i,j,k,h) of $1,2,3,4$ denote:

   - P_i – the orthogonal projection of A_i on $A_jA_kA_h$;
   - B_{ij} – the midpoint of the edge A_iA_j;
   - C_{ij} – the midpoint of segment P_iP_j;
   - β_{ij} – the plane $B_{ij}P_hP_k$;
   - δ_{ij} – the plane $B_{ij}P_iP_j$;
   - α_{ij} – the plane through C_{ij} orthogonal to A_kA_h;
   - γ_{ij} – the plane through C_{ij} orthogonal to A_iA_j;

   Prove that if the points P_i are not in a plane, then the following sets of planes are concurrent: (a) α_{ij}; (b) β_{ij}; (c) γ_{ij}; (d) δ_{ij}.

3. Prove the following identity for every $n \in \mathbb{N}$:

$$\sum_{\substack{j+h=n \\ j \geq h}} \frac{(-1)^h 2^{j-h} \binom{j}{h}}{j} = \frac{2}{n}.$$

4. A sequence (a_n) of positive integers satisfies $(a_m,a_n) = a_{(m,n)}$ for all m,n. Prove that there is a unique sequence (b_n) of positive integers such that

$$a_n = \prod_{d|n} b_d.$$

First Test for IMO

1. Let $A_1,\dots,A_5$ be five points in a plane such that the area of each triangle $A_iA_jA_k$ $(1 \leq i < j < k \leq 5)$ is greater than 3. Prove that one of the triangles $A_iA_jA_k$ has an area greater than 4.

2. The sequence (a_n) is defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n + k$, where k is a positive integer. Find the least k for which a_{1991} and 1991 are not coprime.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

3. Let C be a coloring of all edges and diagonals of a convex $n-$gon in red and blue (in Romanian, *rosu* and *albastru*). Denote by $q_r(C)$ (resp. $q_a(C)$) the number of quadrilaterals having all its edges and diagonals red (resp. blue). Prove:

$$\min_C (q_r(C)+q_a(C)) \leq \frac{1}{32}\binom{n}{4}.$$

4. Let $\mathscr{S}$ be the set of all polygonal areas in a plane. Prove that there is a function $f : \mathscr{S} \to (0,1)$ which satisfies

$$f(S_1 \cup S_2) = f(S_1) + f(S_2)$$

for any $S_1, S_2 \in \mathscr{S}$ which have common points only on their borders.

Second Test for IMO

1. In a triangle $A_1A_2A_3$, the excribed circles corresponding to sides A_2A_3, A_3A_1, A_1A_2 touch these sides at T_1, T_2, T_3, respectively. If H_1, H_2, H_3 are the orthocenters of triangles $A_1T_2T_3$, $A_2T_3T_1$, $A_3T_1T_2$, respectively, prove that lines H_1T_1, H_2T_2, H_3T_3 are concurrent.

2. Let $n \geq 3$ be an integer. A finite number of disjoint arcs with the total sum of length $1-\dfrac{1}{n}$ are given on a circle of perimeter 1. Prove that there is a regular n-gon whose all vertices lie on the considered arcs.

3. Let $x_1, x_2, \dots, x_n$ be positive real numbers with the sum 1. Prove that

$$x_1^2x_2^2\cdots x_n^2 + x_2^2x_3^2\cdots x_{n+1}^2 + \cdots + x_{2n}^2x_1^2\cdots x_{n-1}^2 < \frac{1}{n^{2n}}.$$

4. Let n, a, b be integers with $n \geq 2$ and $a \notin \{0,1\}$ and let $u(x) = ax + b$ be the function defined on integers. Show that there are infinitely many functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f^n(x) = \underbrace{f(f(\cdots f}_{n}(x)\cdots)) = u(x) \qquad \text{for all } x.$$

If $a = 1$, show that there is a b for which there is no f with $f^n(x) \equiv u(x)$.

Third Test for IMO

1. The diagonals of a pentagon $ABCDE$ determine another pentagon $MNPQR$. If $MNPQR$ and $ABCDE$ are similar, must $ABCDE$ be regular?

2. Let $a_1 < a_2 < \cdots < a_n$ be positive integers. We say that a coloring of the set $\mathbb{Z}$ is periodic with period t, if for any integer x exactly one of the numbers $x+a_1, x+a_2, \dots, x+a_n$ is blue. If there is a periodic coloring with period t, prove that n divides t.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com