Romanian Team Selection Tests 1990

Selection Test for Balkan MO

1. Let $f:\mathbb{N}\to\mathbb{N}$ be a function such that the set $\{k \mid f(k)<k\}$ is finite. Prove that the set
$$\{k \mid g(f(k))\geq k\}$$
is infinite for all functions $g:\mathbb{N}\to\mathbb{N}$.

2. If a,b,c are sides of a triangle of circumradius R, prove the inequality
$$\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c}\geq 3R\sqrt{3}.$$

3. Let n be a positive integer. Prove that the least common multiple of numbers $\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n}$ is equal to the least common multiple of numbers $1,2,\dots,n$ if and only if $n+1$ is a prime.

4. Let M be a point on the edge CD of a tetrahedron $ABCD$ such that the tetrahedra $ABCM$ and $ABDM$ have the same total areas. We denote by π_{AB} the plane ABM. Planes $\pi_{AC},\dots,\pi_{CD}$ are analogously defined. Prove that the six planes $\pi_{AB},\dots,\pi_{CD}$ are concurrent in a certain point N, and show that N is symmetric to the incenter I with respect to the barycenter G.

First Test for IMO

1. Let a,b,n be positive integers such that $(a,b)=1$. Prove that if (x,y) is a solution of the equation $ax+by=a^n+b^n$, then
$$\left[\frac{x}{b}\right]+\left[\frac{y}{a}\right]=\left[\frac{a^{n-1}}{b}\right]+\left[\frac{b^{n-1}}{a}\right].$$

2. Prove the following inequality for all positive integers m,n:
$$\sum_{k=0}^{n}\binom{m+k}{k}2^{n-k}+\sum_{k=0}^{m}\binom{n+k}{k}2^{m-k}=2^{m+n+1}.$$

3. Find all polynomials $P(x)$ such that $2P(2x^2-1)=P(x)^2-1$ for all x.

4. The six faces of a hexahedron are quadrilaterals. Prove that if seven its vertices lie on a sphere, then the eighth vertex also lies on the sphere.

Second Test for IMO

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1. Let O be the circumcenter of an acute triangle ABC and R be its circumcenter. Consider the disks having OA, OB, OC as diameters, and let Δ be the set of points in the plane belonging to at least two of the disks. Prove that the area of Δ is greater than $R^2/8$.

2. Prove that there are infinitely many n's for which there exists a partition of $\{1,2,\dots,3n\}$ into subsets $\{a_1,\dots,a_n\}$, $\{b_1,\dots,b_n\}$, $\{c_1,\dots,c_n\}$ such that $a_i+b_i=c_i$ for all i, and prove that there are infinitely many n's for which there is no such partition.

3. The sequence x_n is df=efined by $x_1=1$ and $x_{n+1}=\dfrac{x_n}{n}+\dfrac{n}{x_n}$. Prove that this sequence is increasing and that $[x_n^2]=n$ for each n.

4. For a set S of n points, let $d_1>d_2>\cdots>d_k>\cdots$ be the distances between the points. A function $f_k : S\to\mathbb{N}$ is called a *coloring function* if, for any pair M,N of points in S with $MN\geq d_k$, it takes the value $f_k(M)+f_k(N)$ at some point. Prove that for each $m\in\mathbb{N}$ there are positive integers n,k and a set S of n points such that every coloring function f_k of S satisfies $|f_k(S)|\geq m$.

Third Test for IMO

1. The distance between any two of six given points in the plane is at least1. Prove that the distance between some two points is at least $\sqrt{\dfrac{5+\sqrt{5}}{2}}$.

2. Suppose that p,q are positive primes such that $q\mid 2^p+3^p$. Prove that $q>p$.

3. In a group of n persons,

   (i) each person is acquainted to exactly k others;

   (ii) any two acquainted persons have exactly l common acquaintances;

   (iii) any two non-acquainted persons have exactly m common acquaintances.

   Prove that $m(n-k-1)=k(k-l-1)$.

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