Romanian Team Selection Tests 1988

Selection Test for Balkan MO
Cluj, April 15

1. Let be given a sphere and a plane α. For a variable point $M \in \alpha$ outside the sphere, consider a circular cone with vertex M which is tangent to the sphere. Find the locus of the centers of the circles of tangency of the cone with the sphere. *(O. Stănăşilă)*

2. Let $OXYZ$ be a trihedral angle with $\angle YOZ = \alpha$, $\angle ZOX = \beta$, $\angle XOY = \gamma$, where $\alpha + \beta + \gamma = \pi$. For any point P inside the trihedral angle let P_1, P_2, P_3 be the projections of P on the three faces. Prove that
$$OP \geq PP_1 + PP_2 + PP_3.$$
(C. Cocea)

3. Consider all regular convex or star polygons with n sides inscribed in a given circle. We say that two such polygons are *equivalent* if one can be obtained from the other by a rotation around the center. How many classes of such polygons exist? *(M. Becheanu)*

4. Prove that for all positive integers $0 < a_1 < a_2 < \cdots < a_n$ it holds that
$$(a_1 + a_2 + \cdots + a_n)^2 \leq a_1^3 + a_2^3 + \cdots + a_n^3.$$
(V. Vâjâitu)

5. The cells of a 11×11 chessboard are colored in three colors. Prove that there exists a rectangle on the board whose four corner cells have the same color. *(I. Tomescu)*

First Test for IMO
Craiova, June 10

1. *Find all $n-$tuples of real numbers $(x_1, x_2, \dots, x_n)$ with the property that each number is the sum of the reciprocals of the remaining numbers. *(M. Becheanu)*

2. Lines d_1, d_2, a circle C with its center on d_1 and a circle C_1 which is tangent to d_1, d_2 and C. Find the locus of the tangent point of C with C_1 as the center of C varies on d_1. *(M. Becheanu)*

3. The positive integer n is given and for all positive integers k, $1 \leq k \leq n$, denote by a_{kn} the number of all ordered sequences $(i_1, i_2, \dots, i_k)$ of positive integers which verify the two conditions:

   (i) $1 \leq i_1 < i_2 < \cdots < i_k \leq n$;
   (ii) $i_{s+1} - i_s \equiv 1 \pmod 2$ for all $s = 1, 2, \dots, k-1$.

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Compute the number of $a(n) = \sum_{k=1}^{n} a_{kn}$. *(I. Tomescu)*

4. Show that for all positive integers n the number $\prod_{k=1}^{n} k^{2k-n-1}$ is also an integer number. *(L. Panaitopol)*

Second Test for IMO
Craiova, June 11

1. Let $p > 2$ be a prime number. Find the least positive integer a which can be represented as
$$a = (x-1)f(x) + (x^{p-1} + \cdots + x + 1)g(x)$$
for some polynomials f and g with integer coefficients. *(M. Becheanu)*

2. Let x, y, z be real numbers with the sum 0. Prove that
$$|\cos x| + |\cos y| + |\cos z| \geq 1.$$
(V. Vâjâitu, B. Enescu)

3. Four circles with the centers at the four vertices of a square are constructed, so that the sum of their areas equals the area of the square. An arbitrary point within each circle is taken. Prove that these four points are the vertices of a convex quadrilateral. *(L. Panaitopol)*

4. Let a be a positive integer. The sequence (x_n) is defined by $x_1 = 1$, $x_2 = a$ and $x_{n+2} = ax_{n+1} + 1$ for all $n \geq 1$. Prove that (x,y) is a solution of the equation
$$|y^2 - axy - x^2| = 1$$
if and only if there is an index k for which $(x,y) = (x_k, x_{k+1})$.
(Ş. Buzeţeanu)

Third Test for IMO
Craiova, June 12

1. Let $\mathcal{T}$ denote the set of all plane triangles. The function $f : \mathcal{T} \to \mathbb{R}^+$ is defined by
$$f(ABC) = \min\left(\frac{b}{a}, \frac{c}{b}\right),$$
where $a \leq b \leq c$ are sides of triangle ABC. Find the set of values of f.

2. Let be given an interval $[a,b]$ not containing any integer. Prove that there exists $N>0$ such that the interval $[Na,Nb]$ is of length greater than $1/6$ and contains no integers.

3. For given finite sets $A_1,A_2,\dots,A_n$, let $d(A_1,\dots,A_n)$ denote the number of elements which appear in an odd number of sets among $A_1,\dots,A_n$. Prove that for any positive integer $k\le n$ the number

$$d(A_1,\dots,A_n)-\sum_{i=1}^{n}|A_i|+2\sum_{i<j}|A_i\cap A_j|-\cdots+(-1)^k2^{k-1}\sum_{i_1<\cdots<i_k}|A_{i_1}\cap\cdots\cap A_{i_k}|$$

is divisible by 2^k. *(I. Tomescu, D. Popescu)*

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