Romanian IMO Team Selection Tests 1987

First Test
Timişoara, June 8

1. Let a, b, c be distinct real numbers with positive sum. Let M be the set of 3×3 matrices whose each line and each column contain all the numbers a, b, c. Find $\max\{\det A \mid A \in M\}$ and the number of matrices which realize this maximum value. *(M. Becheanu)*

2. Find all positive integers A which can be represented in the form
$$A = \left(m - \frac{1}{n}\right)\left(n - \frac{1}{p}\right)\left(p - \frac{1}{m}\right),$$
where $m \geq n \geq p \geq 1$ are integers. *(I. Bogdan)*

3. Find the maximum possible number of elements of a subset $B \subset A = \{1, 2, \dots, n\}$ such that, for any $x, y \in B$, $x - y$ does not divide $x + y$. *(M. Lascu, D. Miheţ)*

4. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with real coefficients, where $n = \deg P$ is even. Suppose that

   (i) $a_0 > 0$, $a_n > 0$.

   (ii) $a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \leq \dfrac{4 \min(a_0^2, a_n^2)}{n-1}$.

   Prove that $P(x) \geq 0$ for all real x. *(L. Panaitopol)*

Second Test
Timişoara, June 9

1. Find the least number n for which there exist permutations $\alpha, \beta, \gamma, \delta$ of the set $A = \{1, 2, \dots, n\}$ $(n \geq 2)$ with the property
$$\sum_{i=1}^{n} \alpha(i)\beta(i) = 1.9 \sum_{i=1}^{n} \gamma(i)\delta(i).$$
(M. Chiriţă)

2. The plane is tiled with congruent regular hexagons. Prove that no four vertices of these hexagons are vertices of a square. *(G. Nagy)*

3. Prove that there is no integer $n \geq 2$ for which $\dfrac{3^n - 2^n}{n}$ is an integer. *(L. Panaitopol)*

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4. Let $ABCD$ be a square of side a. Segments AE, CF with $AE = a$ and $CF = b$ $(a < b < a\sqrt{3})$, perpendicular to the plane of the square, are constructed on the same side of the plane. If $\mathscr{K}$ denotes the set of interior points of the square, determine

$$\min_{M\in\mathscr{K}} \max(EM, FM) \quad \text{and} \quad \max_{M\in\mathscr{K}} \min(EM, FM).$$

(O. Stănăşilă)

Third Test

Timişoara, June 10

1. Prove that for all real numbers $\alpha_1, \dots, \alpha_n$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} ij\cos(\alpha_i - \alpha_j) \geq 0.$$

(O. Stănăşilă)

2. Suppose a, b, c are integers such that $a+b+c \mid a^2+b^2+c^2$. Show that $a+b+c \mid a^n+b^n+c^n$ for infinitely many positive integers n.

(L. Panaitopol)

3. For a real number a with $|a| \geq 1$, consider the polynomial $P(x) = x^2+2axy+y^2$. Let $n \geq 2$ be an integer. Consider the system of equations

$$P(x_1, x_2) = P(x_2, x_3) = \cdots = P(x_{n-1}, x_n) = P(x_n, x_1) = 0.$$

We say that two solutions $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ of the system are equivalent if for some real number $t \neq 0$, $x_i = ty_i$ for all i. How many non-equivalent solutions does the system have? *(M. Becheanu)*

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