Romanian IMO Team Selection Tests 1978

First Test – April 9

Time: 4 hours

1. Prove that for any partition of $X = \{1,2,\dots,9\}$ into two subsets, one of the subsets contains distinct elements a,b,c such that $2c = a+b$.

2. Suppose that k,l are positive integers such that $(11m-1,k) = (11m-1,l)$ for every $m \in \mathbb{N}$. Prove that there exists an integer n such that $k = 11^n l$.

3. Let $P(x,y)$ be a polynomial of degree at most 2. If A,B,C,A',B',C' are distinct points in the $xy-$plane at which P vanishes such that A,B,C are not collinear and A',B',C' lie on lines BC,CA,AB respectively, show that P is identically zero.

4. Diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at point O. Prove that if triangles OAB,OBC,OCD,ODA have the same perimeter, then $ABCD$ is a rhombus. What happens if O is some other point inside the quadrilateral?

5. Prove that there is no square with its four vertices on four concentric circles whose radii form an arithmetic progression.

6. Show that there is no polyhedron whose projection on every plane is a nondegenerate triangle.

7. Let P,Q,R be polynomials of degree 3 with real coefficients such that, for every real x, $P(x) \le Q(x) \le R(x)$. Suppose there is a real number a such that $P(a) = R(a)$. Show that $Q = kP + (1-k)R$ for some real number $k \in [0,1]$. What happens if P,Q,R are of degree 4?

8. For any set A, we say that two functions $f,g : A \longrightarrow A$ are *similar* if there exists a bijection $h : A \longrightarrow A$ such that $f \circ h = h \circ g$.

   (a) If A has three elements, construct functions $f_1,\dots,f_k : A \to A$, every two of which are similar, such that every function $f : A \to A$ is similar to one of them.

   (b) If $A = \mathbb{R}$, show that the functions $\sin x$ and $-\sin x$ are similar.

9. A sequence (x_n) of real numbers satisfies $x_0 = \alpha > 1$ and $x_{n+1}(x_n - [x_n]) = 1$ for each $n \ge 1$. Prove that if (x_n) is periodic, then α is a root of a quadratic equation. Study the converse.

Second Test – April 10

Time: 3 hours

1. Consider the net of integer points in the $xy-$plane. We want to associate $a_{h,k}$ to each point (h,k) $(h,k \in \mathbb{Z})$ so that

$$a_{h,k} = \frac{1}{4}(a_{h-1,k} + a_{h+1,k} + a_{h,k-1} + a_{h,k+1}) \qquad \text{for all } h,k.$$

   (a) Prove that it is possible to associate $a_{h,k}$ which are not all equal.

   (b) If so, prove that then $a_{h,k}$ are not bounded from either side.

2. Consider the function $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n^2$. Prove that there exists a function $F : \mathbb{N} \to \mathbb{N}$ such that $F(F(n)) = f(n)$.

3. Let $A_1, A_2, \dots, A_{3n}$ $(n \in \mathbb{N})$ be points such that $A_1A_2A_3$ an equilateral triangle, and for $1 \le k < n$, $A_{3k+1}, A_{3k+2}, A_{3k+3}$ are the midpoints of the sides of $\triangle A_{3k-2}A_{3k-1}A_{3k}$. Each of the $3n$ points is colored in red or blue.

   (a) Prove that for $n \ge 7$ there exists a monochromatic isosceles trapeziod.

   (b) What if $n = 6$?

4. Let $\mathscr{M}$ be a set of $3n$ points whose diameter is 1. Prove that:

   (a) among any four points some two are on a distance at most $1/\sqrt{2}$;

   (b) for $n = 2$, there is a configuration of 6 points with 12 of 15 distances in the interval $(1-\varepsilon, 1]$, where $\varepsilon > 0$ is any given number, but it is not possible that 13 of the distances are in the interval $(1/\sqrt{2}, 1]$;

   (c) there exists a circle of radius at most $\sqrt{3/2}$ which contains $\mathscr{M}$.

   (d) some two points of $\mathscr{M}$ are on a distance not exceeding $\dfrac{4}{3\sqrt{n} - \sqrt{3}}$.

Third Test – June 22

Time: 4 hours

1. In a convex quadrilateral $ABCD$, let A', B' be the orthogonal projections of A, B to CD respectively.

   (a) Assume $BB' \le AA'$ and $S_{ABCD} = \frac{1}{2}(AB + CD)BB'$. Is $ABCD$ necessarily a trapezoid?

   (b) The same question under the assumption that $\angle BAD$ is obtuse.

2. Points A', B', C' are arbitrarily taken on edges SA, SB, SC respectively of a tetrahedron $SABC$. Let planes ABC and $A'B'C'$ intersect in a line d. If plane $A'B'C'$ rotates around d, prove that lines AA', BB', CC' remain concurrent and find the locus of the intersection points.

3. Let D_1, D_2, D_3 be pairwise skew lines. Through every point $P_2 \in D_2$ there is a unique common secant of these three lines, and it meets D_1 at P_1 and D_3 at P_3.

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(a) Let coordinate systems be introduced on D_2 and D_3 with the origins at O_2 and O_3 respecively. Find a relation between the coordinates x_2 of P_2 and x_3 of P_3.

(b) Show that there exist four pairwise skew lines with exactly two common secants. Also find examples with exactly one and with no common secants.

(c) Let F_1,F_2,F_3,F_4 be any four secants of D_1,D_2,D_3. Prove that F_1,F_2,F_3,F_4 have infinitely many common secants.

4. For every $n \in \mathbb{N}$, solve the equation

$$\sin x \sin 2x \cdots \sin nx + \cos x \cos 2x \cdots \cos nx = 1.$$

5. Find the locus of points M inside an equilateral triangle ABC such that $\angle MBC + \angle MCA + \angle MAB = \pi/2$.

6. (a) Prove that in the set $\{x\sqrt{2}+y\sqrt{3}+z\sqrt{5} \mid x,y,z \in \mathbb{Z} \text{ not all zero}\}$ there are nonzero numbers arbitrarily close to 0.

   (b) If $\sqrt{2},\sqrt{3},\sqrt{5}$ are replaced with their rational approximations a,b,c, show that $xa+yb+zc$ is 0 for infinitely many triples (x,y,z) of integers, but cannot be arbitrarily close to 0.

7. A set $\mathscr{M}$ of points, no three collinear, is given in a Cartesian plane. Consider the assertion (A): The barycenter of any finite subset of $\mathscr{M}$ has integral coordinates.

   (a) Prove that for any $n \geq 1$ there is a set $\mathscr{M}$ of n points in which (A) holds.

   (b) Prove that (A) is false if $\mathscr{M}$ is infinite.

8. Reformulate in set-theoretical language and solve:

   Some boys and girls are on a party. It is known that for every subset M of boys there are at least $|M|$ girls which are acquainted with some boys from M. Prove that every boy can dance with a girl of his acquaintance.

Fourth Test – June 24

Time: 4 hours

1. Show that for every natural number $a \geq 3$ there are an infinity of natural numbers n such that $n \mid a^n - 1$. Does this hold for $a = 2$?

2. A real function defined on a set $\{x_1,\dots,x_k\}$ of real numbers is said to be additive if, whenever $n_1x_1+\cdots+n_kx_k = 0$ for natural numbers n_i, it holds that $n_1 f(x_1)+\cdots+n_k f(x_k) = 0$. Show that for every such function f and all real numbers $y_1,\dots,y_p$ there exists an additive real function F on the set $\{x_1,\dots,x_k,y_1,\dots,y_p\}$ such that $F(x_i) = f(x_i)$ for all i.

3. Let $\{A_1,\dots,A_p\}$ and $\{B_1,\dots,B_p\}$ be two partitions of a finite set M such that, whenever $A_i \cap B_j = \emptyset$, it holds that $|A_i|+|B_j| \geq p$. Show that $|M| \geq \frac{1}{2}(p^2+1)$. Can equality hold?

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4. Let $\mathscr{M}$ be a set of n points in a plane, no three collinear. With every segment joining points in $\mathscr{M}$ we associate 1 or -1. A triangle with vertices in $\mathscr{M}$ is called *negative* if the product of numbers at its sides is -1. If -1 is associated with p segments and n is even (odd), show that the number of negative triangles is even (resp. of the same parity as p).

5. Given a triangle $\triangle$, determine the set of points M inside $\triangle$ for which there exists a line d through M dividing $\triangle$ into two regions R_1, R_2 such that $\sigma_d(R_1) \subset R_2$.

6. Can 20 unit regular tetrahedra be placed inside a unit sphere so that no two have common interior points?

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