# Romanian Team Selection Tests 2008

## First Test

1. Given an integer $n \geq 2$, find all sets $A \subseteq \mathbb{Z}$ with $n$ elements such that the sum of the elements of any nonemtpy subset of $A$ is not divisible by $n+1$.

2. Let $a_i$, $b_i$ be positive real numbers, $i \in \{1,2,\ldots,n\}$, $n \geq 2$ such that $a_i < b_i$ for all $i$. Assume that for all $i$:
$$b_1 + \cdots + b_n < 1 + a_1 + \cdots + a_n.$$
Prove that there exists $c \in \mathbb{R}$ such that for all $i \in \{1,2,\ldots,n\}$ and all $k \in \mathbb{Z}$ the following inequality holds:
$$(a_i + c + k)(b_i + c + k) > 0.$$

3. Let $ABCDEF$ be a convex hexagon with all the sides of length 1. Prove that at least one of the radii of the circumcircles of $\triangle ACE$ and $\triangle BDF$ is greater than or equal to 1.

4. Prove that there exists a set $S$ of $n-2$ points inside a convex $n$-gon $P$, such that inside any triangle determined by three vertices of $P$ there is exactly one point from $S$ (inside or on the edges).

5. Find the greatest common divisor of the numbers:
$$2^{561} - 2, 3^{561} - 3, \cdots, 561^{561} - 561.$$

## Second Test

1. Assume that $n \geq 3$ is an odd integer. Determine the maximal value of
$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \cdots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|},$$
where $x_i$ are positive real numbers from $[0,1]$.

2. Do there exist a sequence of positive integers $1 \leq a_1 < a_2 < a_3 < \cdots$ such that for each $n \in \mathbb{N}$ the set $\{a_k + n : k = 1,2,\ldots\}$ contains finitely many prime numbers?

3. Show that each convex pentagon has a vertex $V$ for which the length of the altitude to the opposite edge $v$ is strictly less than the sum of the altitudes to $v$ from the two vertices adjacent to $V$.

4. Let $G$ be a connected graph with $n$ vertices and $m$ edges such that each edge is contained in at least one triangle. Find the minimum value of $m$.

## Third Test

1. Let $ABC$ be a triangle with $\angle BAC < \angle CAB$. Denote by $D$ and $E$ points on the sides $AC$ and $AB$ such that $\angle ACB = \angle BED$. Let $F$ be a point in the interior of the quadrilateral $BCDE$ such that the circumcircle of $\triangle BCF$ is tangent to the circumcircle of $\triangle DEF$ and the circumcircle of $\triangle BEF$ is tangent to the circumcircle of $\triangle CDF$. Prove that the points $A$, $C$, $E$, $F$ lie on a circle.

2. Let $ABC$ be an acute triangle with orthocenter $H$ and let $X$ be an arbitrary point in its plane. THe circle with diameter $HX$ intersects the lines $AH$ and $AX$ at $A_1$ and $A_2$ respectively. The points $B_1$, $B_2$, $C_1$, $C_2$ are defined analogously. Prove that the lines $A_1A_2$, $B_1B_2$, and $C_1C_2$ are concurrent.

3. Given positive integers $m,n \geq 3$, prove that $2^m - 1 \nmid 3^n - 1$.

4. Let $n \in \mathbb{N}$. A set of people is called *n-balanced* if in any subset of 3 persons there exist at least two who know each other, and in any subsets of $n$ persons there are two who don't know each other. Prove that $n$-balanced set can have at most $(n-1)(n+2)/2$ persons.

## Fourth Test

1. Let $ABCD$ be a convex quadrilateral. Let $O$ be the intersection of the diagonals $AC$ and $BD$, $P$ the intersection of the lines $AB$ and $CD$, and $Q$ the intersection of the lines $BC$ and $DA$. Denote by $R$ the feet of perpendicular from $O$ to the line $PQ$. Prove that the feet of perpendiculars from $O$ to the lines determined by the sides of $ABCD$ belong to a circle.

2. Let $m,n \geq 1$ be two relatively prime integers. For each integer $s$ determine the number of $m$-element sets $A \subseteq \{1,2,\dots,m+n-1\}$ such that
$$\sum_{x\in A} x \equiv s \pmod{n}.$$

3. Let $n \geq 3$ be an integer and let $m \geq 2^{n-1}+1$. Prove that for each family of nonzero distinct subsets $(A_j)_{j=1}^{m}$ of $\{1,2,\dots,n\}$ there exist indeces $i$, $j$, $k$ such that $A_i \cup A_j = A_k$?

## Fifth Test

1. Find all $n \in \mathbb{N}$ for which there exists a permutation $\sigma$ of the set $\{1,2,\dots,n\}$ such that the sets
$$\{|\sigma(k)-k| : k \in \{1,2,\dots,n\}\}$$
has exactly $n$ elements.

2. Denote by $k_a$, $k_b$, $k_c$ the circles whose diameters are the medians $m_a$, $m_b$, $m_c$ of $\triangle ABC$, respectively. If two of these circles are tangent to the incircle of $\triangle ABC$, prove that the third circle is tangent as well.

3. Let $\mathscr{P}$ be a square. For each $n \in \mathbb{N}$ denote by $f(n)$ the maximal number of elements of a partition of $\mathscr{P}$ into rectangles such that each line which is parallel to some side of $\mathscr{P}$ intersects at most $n$ interiors of rectangles. Prove that

$$3 \cdot 2^{n-1} - 2 \le f(n) \le 3^n - 2.$$