Romanian Team Selection Tests 2005

First Test

1. Find all positive integers x, y such that $3^x = 2^x y + 1$.
2. Let n be a positive integer and X be a set of n^2+1 positive integers with the property that every $(n+1)$-element subset of X contains two distinct elements one of which divides the other one. Prove that there are distinct elements $x_1, x_2, \dots, x_n$ of X such that $x_i \mid x_{i+1}$ for $i = 1, \dots, n$.
3. The distance from a point inside a convex polyhedron with n faces to any vertex of the polyhedron is at most 1. Prove that the sum of the distances from the point to the planes containing faces of the polyhedron is less than $n-2$.

Second Test

1. If $n \in \mathbb{N}$, prove that in any convex $(4n+2)$-gon of area 1 there is a diagonals that cuts off a triangle of area at most $\dfrac{1}{6n}$.
2. Let m and n be coprime positive integers with m even and n odd. Prove that the sum
$$\frac{1}{2n} + \sum_{k=1}^{n-1} (-1)^{\left[\frac{mk}{n}\right]} \left\{ \frac{mk}{n} \right\}$$
does not depend on m and n.
3. A sequence (a_n) of real numbers is called *modular* if $a_0 = a$, $a_1 = b \neq a$ with $a, b > 0$ and $a_n = |a_{n+1} - a_{n+2}|$ for every $n \geq 0$. Determine if there is a bounded modular sequence.

Third Test

1. Let $A_0, A_1, \dots, A_5$ be sequentially ordered points on a circle γ. In the sequel, indices are reduced modulo 6. Let, for $k = 0, 1, 2$, the line through A_{2k} parallel to $A_{2k+2}A_{2k+4}$ meet γ again at A'_{2k}, and let the lines $A'_{2k}A_{2k+3}$ and $A_{2k+2}A_{2k+4}$ meet at A'_{2k+3}. Prove that if the lines $A_{2k}A_{2k+3}$ $(k = 0, 1, 2)$ are concurrent, then so are the lines $A_{2k}A'_{2k+3}$ $(k = 0, 1, 2)$.
2. Suppose that D, E, F are points on the sides BC, CA, AB of a triangle ABC, respectively, such that the inradii of the triangles AEF, BFD, CDE are equal to half the inradius of the triangle ABC. Prove that D, E, F are the midpoints of the sides of $\triangle ABC$.
3. Prove that if any two vertices of a plane polygon are on a distance at most 1, then the area of the polygon is less than $\sqrt{3}/2$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

Fourth Test

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which there exists a nonzero real constant a such that $f(a+x) = f(x) - x$ for all x.

2. If $n \geq 0$ is an integer and $p \equiv 7 \pmod 8$ a positive prime number, prove that

$$\sum_{k=1}^{p-1} \left\{ \frac{k^{2^n}}{p} + \frac{1}{2} \right\} = \frac{p-1}{2}.$$

3. Let k be a positive integer. Given a sequence $(c_n)_{n=1}^{\infty}$ of decimal digits, for $n \in \mathbb{N}$ we are allowed to intercalate between c_n and c_{n+1} some k_n digits, where $1 \leq k_n \leq k$. This will yeald the infinite decimal representation of a real number x, $0 \leq x < 1$.

   (a) Prove that if $k \leq 9$, then there exists a sequence (c_n) for which the resulting number x is never rational.

   (b) Prove that for $k \geq 10$ any such sequence can be brought to a rational number x.

4. The edges of a convex polyhedron are oriented in such a way that at each vertex there is an edge going out and an edge going in. Show that there exists a face of the polyhedron whose border represents an oriented cycle.

Fifth Test

1. On a 2004×2004 chessboard, 2004 queens are placed so that no two are attacking each other. Prove that there are two queens such that the rectangle, at whose opposite vertices the two queens stand, has the semi-perimeter 2004. (Of course, it is assumed that each queen is exactly in the center of the square her majesty is occupying.)

2. Given an integer $n \geq 2$, find the smallest real number $m(n)$ such that for any positive numbers $x_1, \dots, x_n$ with the product 1 we have the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i} \leq \sum_{i=1}^{n} x_i^r \quad \text{for all } r \geq m(n).$$

3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ the number $(m^2+n)^2$ is divisible by $f^2(m) + f(n)$.

4. Suppose that only two vertices of a polyhedron are incident with an odd number of edges, and that these two vertices are adjacent. Prove that for any integer $n \geq 3$ there is a face of the polyhedron whose number of sides is not divisible by n.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com