Romanian IMO Team Selection Tests 2004

First Test

Time: 4 hours.

1. Let a quadrilateral have sides a_1, a_2, a_3, a_4 and the semiperimeter s. Prove the inequality
$$\sum_{i=1}^{4} \frac{1}{s+a_i} \le \frac{2}{9} \sum_{1\le i<j\le 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}} .$$
When does equality occur?

2. A finite family of disjoint rectangles of total area 4 is given in the coordinate plane so that the projection of their union onto the horizontal axis is an interval. Prove that there are three points in the union of the rectangles which form a triangle of area at least 1.

3. Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,
$$f(f(n)) \le \frac{n+f(n)}{2} .$$

4. Let $\mathscr{D}$ be a closed disc in the complex plane. Prove that for any $n \in \mathbb{N}$ and complex numbers $z_1, z_2, \dots, z_n \in \mathscr{D}$ there exists $z \in \mathscr{D}$ such that
$$z^n = z_1 z_2 \dots z_n.$$

Second Test

Time: 4 hours.

1. A disk is divided into $2n$ equal sectors. Among these sectors, n are colored in black and n in white. Let us number the white sectors from 1 to n clockwise, and the black ones from 1 to n counterclockwise. Show that there is a half-disc consisting of n sectors which contains all numbers from 1 to n.

2. Find all positive integers that can be written in the form $\dfrac{a^2+ab+b^2}{ab-1}$ for some positive integers a, b not both equal to 1.

3. Given integers a, b, c with b odd, define a sequence (x_n) by $x_0 = 4$, $x_1 = 0$, $x_2 = 2c$, $x_3 = 3b$ and
$$x_{n+3} = ax_{n-1} + bx_n + cx_{n+1}.$$
Prove that $p \mid x_{p^m}$ for all primes p and positive integers m.

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4. A square $ABCD$ lies inside a circle $\mathscr{C}$. Let $\mathscr{C}_A$ be the circle outside the square that is tangent to AB, AD and $\mathscr{C}$. The circles $\mathscr{C}$ and $\mathscr{C}_A$ meet at A_1. Points B_1, C_1, D_1 are similarly defined. Show that lines AA_1, BB_1, CC_1, DD_1 are concurrent.

Third Test

Time: 4 hours.

1. Let S be a finite set with $|S| \geq 2$. Suppose $A_1, A_2, \dots, A_{101}$ are subsets of S are such that the union of any 50 of them has at least $50|S|/51$ elements. Prove that among these subsets there exist three (distinct), any two of which have a non-empty intersection.

2. Prove that for all positive integers m, n with m odd it holds that

$$3^m n \mid \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

3. In a non-isosceles triangle ABC, let I be the incenter and A', B', C' be the projections of I onto BC, CA, AB, respectively. Lines AA', BB', CC' meet at point P, lines BC and $B'C'$ meet at N, and AC and $A'C'$ meet at M. Prove that $IP \perp MN$.

4. Let $a_1, a_2, \dots, a_n$ be real numbers, where $n \geq 2$. Prove that for any subset S of $\{1, 2, \dots, n\}$ we have

$$\left(\sum_{i \in S} a_i\right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + \cdots + a_j)^2.$$

Fourth Test

Time: 4 hours.

1. Let $m, n \in \mathbb{N}$, $m > 1$. Suppose that $a^m - 1$ is divisible by n for all a coprime to n. Prove that

$$n \leq 4m(2^m - 1).$$

2. Let O be a point in the plane of triangle ABC, and $\mathscr{C}$ be a circle passing through O. Let $\mathscr{O}$ intersect lines OA, OB, OC at P, Q, R, and circles BOc, COA, AOB at K, L, M, respectively. Show that the lines PK, QL and RM are concurrent.

3. Faces of a polyhedron with n faces are colored in black and white so that no two black faces have a common point. Prove that the number of edges shared by two white faces is at least $n - 2$.

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Fifth Test

Time: 4 hours.

1. Let be given circles K_1, K_2, K_3 of radii R_1, R_2, R_3 through a point O, and let A, B, C be the other points of intersection of the pairs of circles (K_2, K_3), (K_3, K_1) and (K_1, K_2), respectively. Suppose that O is inside the triangle ABC and let AO, BO, CO meet the opposite sides of the triangle at A_1, B_1, C_1, respectively. If R is the circumradius of ABC, prove that

$$R \leq \frac{OA_1}{AA_1}R_1 + \frac{OB_1}{BB_1}R_2 + \frac{OC_1}{CC_1}R_3.$$

2. An operation of an $m \times n$ chessboard consists of the following:

   (i) choose several unmarked cells, no two in the same row or column;

   (ii) mark these squares with 1;

   (iii) mark with 0 every unmarked square that is in the same row or column with a square marked with 1.

   A game consists of performing these operations as long as it is possible. What is the maximum possible sum of the numbers on the board at the end of a game?

3. Let p be a prime number and $f(x) = \sum_{i=1}^{p-1} a_i x^{i-1}$ be a polynomial with $a_i = 1$ if i is a square modulo p and $a_i = -1$ otherwise.

   (a) Prove that $f(x) \equiv x - 1$ modulo $(x-1)^2$ if $p \equiv 3 \pmod 4$;

   (b) Prove that $f(x) \equiv 0$ modulo $(x-1)^2$ if $p \equiv 5 \pmod 8$.

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