Romanian IMO Team Selection Tests 2003

First Test – April 23, 2003.

Time: 4 hours.
Each problem is worth 7 points.

1. Let a sequence $\{a_n\}$ $(n \in \mathbb{N})$ of real numbers be defined by $a_1 = 1/2$ and, for each positive integer n,
$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$
Prove that for every $n \in \mathbb{N}$ it holds that $a_1 + a_2 + \cdots + a_n < 1$.

2. Let ABC be a triangle with $\angle BAC = 60^{circ}$. Suppose that there exists a point P inside the triangle such that $PA = 1$, $PB = 2$ and $PC = 3$. Find the maximum possible area of $\triangle ABC$.

3. Let n,k be positive integers such that $n^k > (k+1)!$. Consider the set
$$M = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \{1, 2, \ldots, k\} \text{ for } i = 1, \ldots, n\}.$$
Prove that in every $(k+1)!+1$-element subset A of M there exist two elements $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ such that $(k+1)! \mid (b_1 - a_1)(b_2 - a_2) \cdots (b_k - a_k)$.

Second Test – April 24, 2003.

Time: 4 hours.
Each problem is worth 7 points.

1. Prove that among the elements the sequence $\left[n\sqrt{2003}\right]$ one can find a geometric progression of an arbitrary length and with arbitrarily large ratio.

2. Let f be an irreducible monic polynomial with integer coefficients, such that $|f(0)|$ is not a perfect square. Prove that the polynomial $g(x) = f(x^2)$ is also irreducible over non-constant polynomials with integer coefficients.

3. At a math contest $2n$ students take part ($n \in \mathbb{N}$). Each student submits a problem to the jury, which thereafter gives each student one of the $2n$ submitted problems. We call a distribution of the problems *fair* if there exist n students that received problems from the other n participants. Prove that the number of fair distributions is a perfect square.

Third Test – May 24, 2003.

Time: 4 hours.
Each problem is worth 7 points.

1. Find all integers a,b,m,n, where $m>n>0$, such that the polynomial $f(x)=x^n+ax+b$ divides the polynomial $g(x)=x^m+ax+b$.

2. Let ω_1 and ω_2 be two externally tangent circles with radii r_1 and $r_2>r_1$ respectively. Let their external common tangent t_1 meet ω_1 and ω_2 at points A and D respectively. The line t_2 is parallel to t_1 and tangent to ω_1 and intersects ω_2 at points E and F. The line t_3 through D intersects t_2 and ω_2 again at B and C respectively. Prove that the circumcircle of triangle ABC is tangent to line t_1.

3. Let $n\geq 3$ be a positive integer. In the cells of a $n\times n$ matrix there are placed n^2 positive real numbers with sum n^3. Prove that there exist four elements which form a 2×2 square with sides parallel to the sides of the matrix, and whose sum is greater than $3n$.

Fourth Test – May 25, 2003.

Time: 4 hours.
Each problem is worth 7 points.

1. Let P be the set of all primes and M be a subset of P, having at least three elements, with the following property: For any proper subset A of M, all the prime factors of the number $\prod_{p\in A} p-1$ are in M. Prove that $M=P$.

2. Let A,B,C,D be points in a square of side 6, such that the distance between any two of them is at least 5. Prove that $ABCD$ is a convex quadrilateral of area greater than 21.

3. Consider all words consisting of letters from the alphabet $\{a,b,c,d\}$. A word is said to be *complicated* if it contains two consecutive identical groups of letters (for example, *caab* and *cababdc* are complicated, while *abcab* is not); otherwise it is said to be *simple*. Prove that there are more than 2^n simple words of length n.

Fifth Test – June 19, 2003.

Time: 4 hours.
Each problem is worth 7 points.

1. A parliament consists of n deputies. The deputies form 10 parties and 10 committees, such that each deputy belongs to exactly one party and one committee. Find the least n for which one can label the parties and the committees with numbers from 1 to 10 so that there exist at least 11 deputies, each of which belongs to a party and a committee which are labelled with the same number.

2. Suppose $ABCD$ is a rhombus of side 1 and M and N points on the sides BC and CD respectively such that $CM+MN+NC=2$ and $\angle BAD = 2\angle MAN$. Find the angles of the rhombus.

3. We say that a point $A(x,y)$ is an *integer point* if both x and y are integers. Denote $O(0,0)$. An integer point A is said to be *invisible* if the segment OA contains at least one integer point. Given any $n \in \mathbb{N}$, prove that there exists a square of side n whose all interior integer points are invisible.

Sixth Test – June 20, 2003.

Time: 4 hours.
Each problem is worth 7 points.

1. In a convex hexagon $ABCDEF$, points A',B',C',D',E',F' are the midpoints of segments AB, BC, CD, DE, EF, FA respectively. Given the areas of the triangles ABC', BCD', CDE', DEF', EFA', FAB', find the area of the hexagon.

2. A permutation σ of the set $\{1,2,\dots,n\}$ is called *straight* if for each $k=1,\dots,n-1$, $\sigma(k)-\sigma(k+1)\le 2$. Find the smallest $n\in\mathbb{N}$ for which there exist at least 2003 straight permutations.

3. Let $d(n)$ denote the sum of decimal digits of a positive integer n. Prove that for each $k\in\mathbb{N}$ there exists a positive integer m such that the equation $x+d(x)=m$ has exactly k solutions in $\mathbb{N}$.

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