# Romanian IMO Team Selection Tests 2002

*First Test – March 21, 2002.*

Time: 4 hours

1. Find all pairs $A, B$ of sets satisfying the following conditions:

   (i) $A \cup B = \mathbb{Z}$;

   (ii) if $x \in A$ then $x - 1 \in B$;

   (iii) if $x, y \in B$ then $x + y \in A$.

2. The sequence $(a_n)$ is defined by

$$a_0 = a_1 = 1 \quad \text{and} \quad a_{n+1} = 14a_n - a_{n-1} \text{ for all } n \geq 1.$$

   Prove that $2a_n - 1$ is a perfect square for any $n \geq 0$.

3. In an acute triangle $ABC$, let $M, N$ be the midpoints of $AB$ and $AC$ respectively, $P$ be the projection of $N$ on $BC$ and $A_1$ be the midpoint of $MP$. Points $B_1$ and $C_1$ are constructed similarly. Prove that if $AA_1, BB_1$ and $CC_1$ are concurrent then $\triangle ABC$ is isosceles.

4. For any $n \in \mathbb{N}$ let $f(n)$ be the number of choices of signs $+/-$ in the expression $E = \pm 1 \pm 2 \pm \cdots \pm n$ which yield the value $E = 0$. Prove that:

   (a) if $n \equiv 1, 2 \pmod 4$ then $f(n) = 0$;

   (b) if $n \equiv 0, 3 \pmod 4$ then

$$\sqrt{2}^{n-2} \leq f(n) < 2^n - 2^{[n/2]+1}.$$

*Second Test – April 13, 2002.*

Time: 4 hours

1. Let $M$ and $N$ be points in the interior of a square $ABCD$ such that the line $MN$ contains no vertex of the square. Denote by $s(M, N)$ the smallest area of a triangle with vertices in the set $\{A, B, C, D, M, N\}$. Find the smallest real number $k$ such that for any such points $M, N$ it holds that $s(M, N) \leq k$.

2. Assume that $P$ and $Q$ are polynomials with coefficients in the set $\{1, 2002\}$ such that $P$ divides $Q$, prove that then $\deg P + 1$ divides $\deg Q + 1$.

3. Given positive real numbers $a, b$, define $x_n$ $(n \in \mathbb{N})$ as the sum of digits of $[an + b]$. Prove that there exists a positive integer which occurs in the sequence infinitely often.

4. At an international conference there are four official languages. Any two participants can talk to each other in at least one of the official languages. Prove that there is a language which is spoken by at least 60 percents of the participants.

*Third Test – April 14, 2002.*

Time: 4 hours

1. A pentagon $ABCD$ inscribed in a circle with center $O$ has angles $\angle B = \angle C = 120^\circ$, $\angle D = 130^\circ$, $\angle E = 100^\circ$. Prove that the intersection point of $BD$ and $CE$ lies on $AO$.

2. Let $a_1, a_2, \ldots, a_n$ be positive real numbers ($n \geq 3$) such that $a_1^2 + \cdots + a_n^2 = 1$. Prove the inequality
$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}\left(a_1\sqrt{a_1} + \cdots + a_n\sqrt{a_n}\right)^2.$$

3. For an even positive integer $n$, let $S$ denote the set of natural numbers $a$, $1 < a < n$ for which $a^{a-1} - 1$ is divisible by $n$. If $S = \{n-1\}$, prove that $n = 2p$ for some prime number $p$.

4. Suppose $f : \mathbb{Z} \to \{1, 2, \ldots, n\}$ is a function such that $f(x) \neq f(y)$ whenever $|x-y|$ is 2, 3 or 5. Prove that $n \geq 3$.

*Fourth Test – June 1, 2002.*

Time: 4 hours

1. Given $p_0, p_1 \in \mathbb{N}$, define $p_{n+2}$ ($n \geq 0$) inductively to be the smallest prime divisor of $p_n + p_{n+1}$. Prove that the real number whose decimal representation is given by $x = \overline{0.p_0 p_1 p_2 \ldots}$ is rational.

2. Consider a unit square $A_1A_2A_3A_4$. Determine the smallest real number $a > 0$ with the following property: For any positive reals $r_1, r_2, r_3, r_4$ with sum $a$ there exist points $X_i$ in the plane satisfying $X_iA_i \leq r_i$ ($1 \leq i \leq 4$) such that one of the triangles with vertices in $X_1, X_2, X_3, X_4$ is equilateral.

3. In a parliament there are several parties, and each member of the parliament has a constant absolute rating. Within a party, each member has a relative rating which is equal to the ratio of his/her rating to the sum of all the ratings in the party. A member of the parliament may change the party only if that would increase his/her relative rating. Prove that after finitely many changes of parties no more changes will be possible.

*Fifth Test – June 2, 2002.*

Time: 4 hours

1. Let $m$ and $n$ be positive integers, not of the same parity, such that $m < n < 5m$. Show that the set $\{1, 2, \dots, 4mn\}$ can be partitioned into pairs of numbers so that the sum in each pair is a square.

2. Let a triangle $ABC$ with $AB < AC \neq BC$ be inscribed in a circle $\mathscr{C}$. The tangent at $A$ to $\mathscr{C}$ intersects $BC$ at $D$. The circle tangent to segments $BD, AD$ and circle $\mathscr{C}$ meets $BC$ at $M$. Prove that $\angle DAM = \angle MAB$ if and only if $AC = CM$.

3. We are given $np$ cards. In each of $n$ colors exactly $p$ cards, numbered $1, 2, \dots, p$, are colored. There are $n$ players playing the following game. Each of them initially receives $p$ cards. The game is glayed in $p$ rounds after the following rules:

   (i) In each round he first player puts down a card; every other player thereafter puts down a card of the same color if he/she has any, and any card otherwise.

   (ii) In each round, the player who put down the card of the initial color which is numbered with the biggest number wins the round.

   (iii) The player who wins a round starts the next round.

   (iv) The first round is started by a random player and after each round the cards player will be taken out of the game.

   Assume that all cards numbered 1 won the rounds in which they were put down. Prove that $p \geq 2n$.