Romanian IMO Team Selection Tests 2001

First Test

Time: 4 hours

1. If complex numbers a, b, c satisfy

$$(a+b)(a+c)=b, \qquad (b+c)(b+a)=c, \qquad (c+a)(c+b)=a,$$

prove that they are real.

2. (a) Let f, g be injective functions from $\mathbb{Z}$ to itself. Prove that the function $h = fg$ is not surjective.

   (b) Given a surjective function $f : \mathbb{Z} \to \mathbb{Z}$, prove that there exist two surjective functions $g, h : \mathbb{Z} \to \mathbb{Z}$ such that $f = gh$.

3. Let a, b, c be sides of a triangle. Prove that

$$(-a+b+c)(a-b+c)+(a-b+c)(a+b-c)+$$
$$+(a+b-c)(-a+b+c) \le \sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

4. Three schools have 200 students each, and every student knows at least one student from each of the other two schools (acquaintances are mutual). Assume there exists a set E of 300 of these students such that for any school S and any two students $x, y \in E$, x and y have distinct numbers of acquaintances in school S. Prove that one can find three students from three distinct schools who know each other.

Second Test

Time: 4 hours

1. Determine all real polynomials P such that

$$P(x)P(2x^2-1) = P(x^2)P(2x-1)$$

holds for all x.

2. Let be given a circle and a square $ABCD$ whose vertices are all in the exterior of the circle. Let AA', BB', CC', DD' be tangents from A, B, C, D to the circle. Consider a circumscribable quadrilateral p whose consecutive side lengths are equal to AA', BB', CC', DD'. Prove that p has an axis of symmetry.

3. Find the least $n \in \mathbb{N}$ such that among any n rays in space there exist two which form an acute angle.

4. Prove that the set of positive integers which cannot be writen as a sum of distinct squares is finite.

Third Test

Time: 4 hours

1. Let n be a positive integer and $f(x) = a_m x^m + \cdots + a_1 x + a_0$ ($m \geq 1$) be a polynomial satisfying:

   (a) for $i = 2, 3, \ldots, m$, a_i is divisible by all prime divisors of n;

   (b) a_1 and n are coprime.

   Prove that for any positive integer k there is a positive integer c such that $f(c)$ is divisible by n^k.

2. We are given a finite set od open disks within a unit square such that the sum of their areas is 1 and the sum of their radii is at least $10/\pi$. Prove that one can draw a disk of radius less than $1/10$ inside the square which intersects at least five of the given disks. (The problem is false, but nevertheless enjoy it)

3. Let $p, q \in \mathbb{N}$ be coprime. A set S of nonnegative integers is called *ideal* if:

   (a) $0 \in S$;

   (b) $n \in S \Rightarrow n + p, n + q \in S$.

   How many ideals are there?

Fourth Test

Time: 4.5 hours

1. Determine all pairs (m, n) of positive integers such that $a^n - 1$ is divisible by m for all $a = 1, 2, \ldots, n$.

2. Prove that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y > 0$,

$$f(x+y) \geq f(x) + yf(f(x)).$$

3. Let ABC be an acute triangle and $\mathscr{C}$ be its circumcircle. The tangents from A and B to $\mathscr{C}$ meet the tangent from C to $\mathscr{C}$ in points D and E respectively. Let AE meet BC at P and BD meet AC at R. Points Q and S are respectively the midpoints of AP and BR. Prove that $\angle ABQ = \angle BAS$.

4. Consider a conver polyhedron $\mathscr{P}$ with vertices $V_1, \ldots, V_p$. Vertices a and b are said to be *neighbors* if they belong to the same face of $\mathscr{P}$. To each vertex V_k we assign a number $v_k(0)$, and construct inductively the sequence $v_k(n)$ ($n \geq 0$) as follows: $v_k(n+1)$ is the average of the $v_j(n)$ for all neighbors V_j of V_k. If all numbers $v_k(n)$ are integers, prove that there exists N such that all $v_k(n)$ are equal for $n \geq N$.

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