Romanian IMO Team Selection Tests 2000

First Test

Time: 4 hours

1. How many functions $f : \{1,2,\dots,n\} \to \{1,2,\dots,5\}$ are there such that for any $k = 1,2,\dots,n-1$ it holds that $|f(k+1) - f(k)| \geq 3$?

2. Suppose $x_1, x_2, \dots, x_{2n}$ are real numbers such that $|x_{i+1} - x_i| \leq 1$ for any i, $1 \leq i \leq 2n-1$. Prove that

$$|x_1| + |x_2| + \cdots + |x_{2n}| + |x_1 + x_2 + \cdots + x_{2n}| \geq n(n+1).$$

3. Prove that for any positive integers n and k one can find integers $a,b,c,d,e > k$ such that

$$n = \pm\binom{a}{3} \pm \binom{b}{3} \pm \binom{c}{3} \pm \binom{d}{3} \pm \binom{e}{3}.$$

4. Suppose that a convex polygon $P_1P_2\dots P_n$ has the property that for any distinct i,j there exists k (distinct from i,j) such that $\angle P_iP_jP_k = 60^\circ$. Prove that $n = 3$.

Second Test

Time: 4 hours

1. Prove that the equation

$$x^3 + y^3 = z^4 - t^2$$

has infinitely many positive integer solutions x,y,z,t such that $(x,y,z,t) = 1$.

2. Let M be an arbitrary point inside a triangle ABC. Prove that

$$\min MA, MB, MC + MA + MB + MC < AB + BC + CA.$$

3. Find all pairs of positive integers (m,n) for which a rectangle $m \times n$ can be tiled with L-trominoes

Third Test

Time: 4 hours

1. Given a positive integer a, find the minimum $k \in \mathbb{N}$ such that $2^{2000} \mid a^k - 1$.

2. In an acute-angled triangle ABC, let M be the midpoint of BC and N be a point in the interior of the triangle such that $\angle NBA = \angle BAM$ and $\angle NCA = \angle CAM$. Prove that $\angle NAB = \angle MAC$.

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3. Let $\mathscr{C}$ be the interior of a circle and $\mathscr{S}$ be the interior of a sphere. Prove that there is no function $f : \mathscr{S} \to \mathscr{C}$ so that

$$d(A,B) \le d(f(A), f(B)) \qquad \text{for any } A,B \in \mathscr{S},$$

where $d(X,Y)$ denotes the distance between points X,Y.

Fourth Test

Time: 4 hours

1. Let P_1 be a regular n-gon, where $n \in \mathbb{N}$. We construct P_2 as the regular n-gon whose vertices are the midpoints of the edges of P_1. Continuing analogously, we obtain regular n-gons $P_3, P_4, \dots, P_m$. For $m \ge n^2 - n + 1$, find the maximum number k such that for any coloring of vertices of $P_1, \dots, P_m$ in k colors there exists a (possible degenerate) isosceles trapezoid whose vertices have the same color.

2. Suppose P, Q are monic complex polynomials such that $P(P(x)) = Q(Q(x))$. Prove that $P = Q$.

3. Show that any positive rational number can be written in the form

$$\frac{a^3+b^3}{c^3+d^3}, \qquad a,b,c,d \in \mathbb{N}.$$

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