49-th Polish Mathematical Olympiad 1997/98

Third Round

April 24–25, 1998

First Day

1. Find all positive integers a,b,c,x,y,z with $a \geq b \geq c$ and $x \geq y \geq z$ which satisfy
$$a+b+c = xyz, \quad x+y+z = abc.$$

2. The Fibonacci sequence (F_n) is given by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Find all pairs (k,m) of integers with $m > k \geq 0$ for which number 1 is a term of the sequence defined by
$$x_0 = \frac{F_k}{F_m}, \quad x_{n+1} = \begin{cases} \dfrac{2x_n - 1}{1 - x_n} & \text{for } x_n \neq 1, \\ 1 & \text{for } x_n \neq 1, \end{cases} \quad n = 0,1,2,\dots$$

3. A convex pentagon $ABCDE$ is the base of a pyramid $ABCDES$. A plane, not passing through any vertex of the pyramid, meets the edges SA,SB,SC,SD,SE in points A',B',C',D',E' respectively. Prove that the intersection points of the diagonals of the quadrilaterals $ABB'A'$, $BCC'B'$, $CDD'C'$, $DEE'D'$, $EAA'E'$ are coplanar.

Second Day

4. Show that the sequence (a_n) defined by a_1 and $a_n = a_{n-1} + a_{[n/2]}$ for $n \geq 2$ contains infinitely many terms divisible by 7.

5. Points D and E lie on the side AB of a triangle ABC and satisfy
$$\frac{AD}{DB} \cdot \frac{AE}{EB} = \frac{AC^2}{CB^2}.$$
Prove that $\angle ACD = \angle BCE$.

6. Consider unit squares in the plane whose vertices have integer coordinates. Let S be the chessboard consisting of all the unit squares lying entirely inside the circle $x^2 + y^2 \leq 1998^2$. In every square of chessboard S number 1 is written. In each move, we may change the signs of all numbers in a row, column or diagonal of S. (A diagonal consists of the squares of S lying on a line which forms an angle of 45° with the axes.) Is it possible to have -1 in exactly one unit square of S after finitely many moves?

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