# 45-th Polish Mathematical Olympiad 1993/94

## Third Round

Warszawa, April 10–11, 1994

*First Day*

1. Determine all triples $(x,y,z)$ of positive rational numbers such that $x+y+z$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ and $xyz$ are all integral.

2. Let be given two parallel lines $k$ and $l$, and a circle not intersecting $k$. Two tangents from a variable point $A \in k$ to the circle intersect the line $l$ at $B$ and $C$. Let $m$ be the line through $A$ and the midpoint of $BC$. Prove that all the lines $m$ (as $A$ varies) have a common point.

3. Given a fixed integer $c \geq 1$, let $a(n)$ be the number of mappings $w$ from the subsets of $\{1,2,\dots,n\}$ to the integers $1,2,\dots,c$ such that

$$w(A\cap B) = \min\{w(A),w(B)\} \qquad \text{for any two subsets } A,B \text{ of } \{1,\dots,n\}.$$

Compute $\lim_{n\to\infty} \sqrt[n]{a(n)}$.

*Second Day*

4. We are given three vessels without the scale: two of them of capacities $m$ and $n$ liters are empty, and the third one of capacity $m+n$ liters is full of water, where $m$ and $n$ are coprime positive integers. Prove that for any $k = 1,2,\dots,m+n-1$, pouring out from one vessel into another, we can obtain exactly $k$ liters of water in the third vessel.

5. Let $O$ be the center of a parallelopiped $A_1A_2\dots A_8$. Prove that

$$4\sum_{i=1}^{8} OA_i^2 \leq \left(\sum_{i=1}^{8} OA_i\right)^2.$$

6. Let $x_1,x_2,\dots,x_n$ $(n\geq 4)$ be different real numbers which satisfy the conditions $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = 1$. Show that there exist four of these numbers, say $a,b,c,d$, such that

$$a+b+c+nabc \leq \sum_{i=1}^{n} x_i^3 \leq a+b+d+nabd.$$



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