# 43-rd Polish Mathematical Olympiad 1991/92

## Third Round

*First Day*

1. Segments $AC$ and $BD$ intersecxt at $P$ so that $PA = PD$ and $PB = PC$. If $O$ is the circumcenter of triangle $PAB$, prove that $OP$ is perpendicular to $CD$.

2. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$,
$$f(x+1) = f(x)+1 \qquad \text{and} \qquad f(x^3) = f(x)^3.$$

3. If $a_1, a_2, \dots, a_r$ are arbitrary real numbers, prove the inequality
$$\sum_{n=1}^{r} \sum_{m=1}^{r} \frac{a_m a_n}{m+n} \geq 0.$$

*Second Day*

4. The sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_0(x) = 8$ and
$$f_{n+1}(x) = \sqrt{x^2 + 6 f_n(x)} \qquad \text{for all } x.$$
For every integer $n \geq 0$, solve the equation $f_n(x) = 2x$.

5. The base of a regular pyramid is a regular $2n-$gon $A_1A_2 \dots A_{2n}$. A sphere passing through the top vertex $S$ intersects the lateral edge $SA_i$ at $B_i$ for $i = 1, 2, \dots, 2n$. Prove that $\sum_{i=1}^{n} SB_{2i-1} = \sum_{i=1}^{n} SB_{2i}$.

6. Prove that for each positive integer $k$, $(k^3)!$ is divisible by $(k!)^{k^2+k+1}$.



The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com