42-nd Polish Mathematical Olympiad 1990/91

Third Round

First Day

1. Prove or disprove that there exist two tetrahedra T_1 and T_2 such that:

   (i) the volume of T_1 is greater than that of T_2;

   (ii) the area of any face of T_1 does not exceed the area of any face of T_2.

2. Let X be the set of all lattice points in the plane (points (x,y) with $x,y \in \mathbb{Z}$). A *path* of length n is a chain $(P_0,P_1,\dots,P_n)$ of points in X such that $P_{i-1}P_i = 1$ for $i = 1,\dots,n$. Let $F(n)$ be the number of distinct paths beginning in P_0 and ending in any point P_n on line $y = 0$. Prove that $F(n) = \binom{2n}{n}$.

3. Define
$$N = \sum_{k=1}^{60} \varepsilon_k k^{k^k},$$
where $\varepsilon_k \in \{-1,1\}$ for each k. Prove that N cannot be the fifth power of an integer.

Second Day

4. On the Cartesian plane consider the set V of all vectors with integer coordinates. Determine all functions $f : V \to \mathbb{R}$ satisfying the conditions:

   (i) $f(v) = 1$ for each of the four vectors $v \in V$ of unit length.

   (ii) $f(v+w) = f(v) + f(w)$ for every two perpendicular vectors $v,w \in V$.

   (Note that the zero vector is perpendicular to every vector.)

5. Two noncongruent circles k_1 and k_2 are exterior to each other. Their common tangents intersect the line through their centers at points A and B. Let P be any point of k_1. Prove that there is a diameter of k_2 with one endpoint on line PA and the other on PB.

6. If x,y,z are real numbers satisfying $x^2+y^2+z^2 = 2$, prove the inequality
$$x+y+z \le 2+xyz$$
and find the conditions for equality.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com