41-st Polish Mathematical Olympiad 1989/90

Third Round

First Day – April 7

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2).$$

2. Let $x_1, x_2, \dots, x_n$ be positive numbers. Prove that
$$\frac{x_1^2}{x_1^2+x_2x_3}+\frac{x_2^2}{x_2^2+x_3x_4}+\cdots+\frac{x_{n-1}^2}{x_{n-1}^2+x_nx_1}+\frac{x_n^2}{x_n^2+x_1x_2}\le n-1.$$

3. On a tournament, any two of the n players played exactly one match (no draws). Prove that it is possible either
   (i) to partition the league in two groups A and B such that everybody in A defeated everybody in B; or
   (ii) to arrange all the players in a chain $x_1, x_2, \dots, x_n, x_1$ in such a way that each player defeated his successor.

Second Day – April 8

4. A triangle whose all sides have length not smaller than 1 is inscribed in a square of side length 1. Prove that the center of the square lies inside the triangle or on its boundary.

5. Suppose that (a_n) is a sequence of positive integers such that $\lim_{n\to\infty}\frac{n}{a_n}=0$. Prove that there exists k such that there are at least 1990 perfect squares between $a_1+a_2+\cdots+a_k$ and $a_1+a_2+\cdots+a_{k+1}$.

6. Prove that for all integers $n>2$, $\sum_{k=0}^{[n/3]}(-1)^k\binom{n}{3k}$ is divisible by 3.

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