39-th Polish Mathematical Olympiad 1987/88

Third Round

First Day

1. The real numbers $x_1, x_2, \ldots, x_n$ from the interval $(0,1)$ satisfy the equality $x_1 + x_2 + \cdots + x_n = m + r$, where m is an integer and $r \in [0,1)$. Prove that
$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq m + r^2.$$

2. For a permutation $\pi = (p_1, p_2, \ldots, p_n)$ of $(1, 2, \ldots, n)$ we define $X(\pi)$ to be the number of indices j such that $p_i < p_j$ for all $i < j$. Find the expected value of $X(\pi)$ if all permutations π are equiprobable.

3. A polygon W has a center of symmetry S. Prove that there is a parallelogram V containing W such that the midpoint of each side of V lies on the boundary of W.

Second Day

4. Let d be a positive integer and let $f : [0,d] \to \mathbb{R}$ be a continuous function with $f(0) = f(d)$. Show that there exists x, $0 \leq x \leq d-1$, such that $f(x) = f(x+1)$.

5. The sequence $(a_n)_{n \in \mathbb{N}}$ is defined by $a_1 = a_2 = a_3 = 1$ and $a_{n+3} = a_{n+2}a_{n+1} + a_n$ for $n \geq 1$. Prove that for any positive integer r there is a positive integer s such that a_s is divisible by r.

6. Determine the largest possible volume of a tetrahedron that lies in the interior of a hemisphere of radius 1.

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