35-th Polish Mathematical Olympiad 1983/84

Third Round

April 6–7, 1984

First Day

1. Find the number of all real functions f which map the sum of n elements into the sum of their images, such that f^{n-1} is a constant function and f^{n-2} is not. Here $f^0(x) = x$ and $f^k = f \circ f^{k-1}$ for $k \geq 1$.

2. Let n be a positive integer. For all $i, j \in \{1, 2, \dots, n\}$ define $a_{j,i} = 1$ if $j = i$ and $a_{j,i} = 0$ otherwise. Also, for $i = n+1, \dots, 2n$ and $j = 1, \dots, n$ define $a_{j,i} = -\frac{1}{n}$. Prove that for any permutation p of the set $\{1, 2, \dots, 2n\}$ the following inequality holds:
$$\sum_{j=1}^{n} \left| \sum_{k=1}^{n} a_{j,p(k)} \right| \geq \frac{n}{2}.$$

3. Let W be a regular octahedron and O be its center. In a plane P containing O circles $k_1(O, r_1)$ and $k_2(O, r_2)$ are chosen so that $k_1 \subset P \cap W \subset k_2$. Prove that $\frac{r_1}{r_2} \leq \frac{\sqrt{3}}{2}$.

Second Day

4. A coin is tossed n times, and the outcome is written in the form $(a_1, a_2, \dots, a_n)$, where $a_i = 1$ or 2 depending on whether the result of the i-th toss is the head or the tail, respectively. Set $b_j = a_1 + a_2 + \cdots + a_j$ for $j = 1, 2, \dots, n$, and let $p(n)$ be the probability that the sequence $b_1, b_2, \dots, b_n$ contains the number n. Express $p(n)$ in terms of $p(n-1)$ and $p(n-2)$.

5. A regular hexagon of side 1 is covered by six unit disks. Prove that none of the vertices of the hexagon is covered by two (or more) discs.

6. Cities $P_1, \dots, P_{1025}$ are connected to each other by airlines $A_1, \dots, A_{10}$ so that for any two distinct cities P_k and P_m there is an airline offering a direct flight between them. Prove that one of the airlines can offer a round trip with an odd number of flights.

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