34-th Polish Mathematical Olympiad 1982/83

Third Round

First Day

1. On the plane are given a convex n-gon $P_1P_2\dots P_n$ and a point Q inside it, not lying on any of its diagonals. Prove that if n is even, then the number of triangles $P_iP_jP_k$ containing the point Q is even.

2. Let be given an irrational number a in the interval $(0,1)$ and a positive integer N. Prove that there exist positive integers p,q,r,s such that

$$\frac{p}{q} < a < \frac{r}{s}, \quad \frac{r}{s} - \frac{p}{q} < \frac{1}{N}, \quad \text{and} \quad rq - ps = 1.$$

3. Consider the following one-player game on an infinite chessboard. If two horizontally or vertically adjacent squares are occupied by a pawn each, and a square on the same line that is adjacent to one of them is empty, then it is allowed to remove the two pawns and place a pawn on the third (empty) square. Prove that if in the initial position all the pawns were forming a rectangle with the number of squares divisible by 3, then it is not possible to end the game with only one pawn left on the board.

Second Day

4. Prove that if natural numbers a,b,c,d satisfy the equality $ab = cd$, then

$$\frac{\gcd(a,c)\gcd(a,d)}{\gcd(a,b,c,d)} = a.$$

5. On the plane are given unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Show that one can choose numbers $c_1, c_2, c_3 \in \{-1,1\}$ such that the length of the vector $c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3$ is at least 2.

6. Prove that if all dihedral angles of a tetrahedron are acute, then all its faces are acute-angled triangles.

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