31-st Polish Mathematical Olympiad 1979/80

Third Round

First Day

1. Compute the area of an octagon inscribed in a circle, whose four sides have length 1 and the other four sides have length 2.

2. Prove that for every n there exists a solution of the equation

$$a^2 + b^2 + c^2 = 3abc$$

in natural numbers a, b, c greater than n.

3. Let k be an integer in the interval $[1,99]$. A fair coin is to be flipped 100 times. Let

$$\varepsilon_j = \begin{cases} 1, & \text{if the } j\text{-th flip is a head;} \\ 2, & \text{if the } j\text{-th flip is a tail.} \end{cases}$$

Let M_k denote the probability that there exists a number i such that $k + \varepsilon_1 + \cdots + \varepsilon_i = 100$. How to choose k so as to maximize the probability M_k?

Second Day

4. Show that for every polynomial W in three variables there exist polynomials U and V such that:

$$\begin{aligned} W(x,y,z) &= U(x,y,z) + V(x,y,z), \\ U(x,y,z) &= U(y,x,z), \\ V(x,y,z) &= -V(x,z,y). \end{aligned}$$

5. In a tetrahedron, the six triangles determined by an edge of the tetrahedron and the midpoint of the opposite edge all have equal area. Prove that the tetrahedron is regular.

6. Prove that for every natural number n we have

$$\sum_{s=n}^{2n} 2^{2n-s} \binom{s}{n} = 2^{2n}.$$

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