30-th Polish Mathematical Olympiad 1978/79

Third Round

First Day

1. Let be given a set $\{r_1, r_2, \dots, r_k\}$ of natural numbers that give distinct remainders when divided by a natural number m. Prove that if $k > m/2$, then for every integer n there exist indices i and j (not necessarily distinct) such that $r_i + r_j - n$ is divisible by m.

2. Prove that the four lines, joining the vertices of a tetrahedron with the incenters of the opposite faces, have a common point if and only if the three products of the lengths of opposite sides are equal.

3. An experiment consists of performing n independent tests. The i-th test is successful with the probability equal to p_i. Let r_k be the probability that exactly k tests succeed. Prove that
$$\sum_{i=1}^{n} p_i = \sum_{k=0}^{n} k r_k.$$

Second Day

4. Let $A > 1$ and $B > 1$ be real numbers and (x_n) be a sequence of numbers in the interval $[1, AB]$. Prove that there exists a sequence (y_n) of numbers in the interval $[1, A]$ such that
$$\frac{x_m}{x_n} \le B \frac{y_m}{y_n} \quad \text{for all } m, n = 1, 2, \dots$$

5. Prove that the product of the sides of a quadrilateral inscribed in a circle with radius 1 does not exceed 4.

6. A polynomial w of degree $n > 1$ has n distinct zeros $x_1, x_2, \dots, x_n$. Prove that:
$$\frac{1}{w'(x_1)} + \frac{1}{w'(x_2)} + \dots + \frac{1}{w'(x_n)} = 0.$$

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