29-th Polish Mathematical Olympiad 1977/78

Third Round

First Day

1. A ray of light reflects from the rays of a given angle. A ray that enters the vertex of the angle is absorbed. Prove that there is a natural number n such that any ray can reflect at most n times.

2. In a coordinate plane, consider the set of points with integer cooedinates at least one of which is not divisible by 4. Prove that these points cannot be partitioned into pairs such that the distance between points in each pair equals 1.

   In other words, an infinite chessboard, whose cells with both coordinates divisible by 4 are cut out, cannot be tiled by dominoes.

3. Prove that if m is a natural number and P,Q,R polynomials of degrees less than m satisfying
$$x^{2m}P(x,y)+y^{2m}Q(x,y)=(x+y)^{2m}R(x,y),$$
   then each of the polynomials is zero.

Second Day

4. Let X be a set of n elements. Prove that the sum of the numbers of elements of sets $A\cap B$, where A and B run over all subsets of X, is equal to $n4^{n-1}$.

5. For a given real number a, define the sequence (a_n) by $a_1=a$ and
$$a_{n+1}=\begin{cases} \dfrac{1}{2}\left(a_n-\dfrac{1}{a_n}\right) & \text{if } a_n\neq 0,\\ 0 & \text{if } a_n=0.\end{cases}$$
   Prove that the sequence (a_n) contains infinitely many nonpositive terms.

6. Prove that if h_1,h_2,h_3,h_4 are the altitudes of a tetrahedron and d_1,d_2,d_3 the distances between the pairs of opposite edges of the tetrahedron, then
$$\frac{1}{h_1^2}+\frac{1}{h_2^2}+\frac{1}{h_3^2}+\frac{1}{h_4^2}=\frac{1}{d_1^2}+\frac{1}{d_2^2}+\frac{1}{d_3^2}.$$

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