

# 27-th Polish Mathematical Olympiad 1975/76

## Third Round

### First Day

1. Is the number  $\sin \frac{\pi}{18} \sin \frac{3\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18} \sin \frac{9\pi}{18}$  rational?
2. Four sequences of real numbers  $(a_n), (b_n), (c_n), (d_n)$  satisfy for all  $n$ ,

$$\begin{aligned}a_{n+1} &= a_n + b_n, & b_{n+1} &= b_n + c_n, \\c_{n+1} &= c_n + d_n, & d_{n+1} &= d_n + a_n.\end{aligned}$$

Prove that if  $a_{k+m} = a_m, b_{k+m} = b_m, c_{k+m} = c_m, d_{k+m} = d_m$  for some  $k \geq 1, n \geq 1$ , then  $a_2 = b_2 = c_2 = d_2 = 0$ .

3. Prove that for each tetrahedron, the three products of pairs of opposite edges are sides of a triangle.

### Second Day

4. The diagonals of some quadrilateral with sides  $a, b, c, d$  are perpendicular. Prove that the diagonals of any other quadrilateral with sides  $a, b, c, d$  also are perpendicular.
5. A trawler is about to fish in territorial waters of a neighboring country, for what he has no licence. Whenever he throws the net, the coast-guard may stop him with the probability  $1/k$ , where  $k$  is a fixed positive integer. Each throw brings him a fish landing of a fixed weight. However, if the coast-guard stops him, they will confiscate his entire fish landing and demand him to leave the country. The trawler plans to throw the net  $n$  times before he returns to territorial waters in his country. Find  $n$  for which his expected profit is maximal.
6. An increasing function  $f : \mathbb{N} \rightarrow \mathbb{R}$  satisfies

$$f(kl) = f(k) + f(l) \quad \text{for all } k, l \in \mathbb{N}.$$

Show that there is a real number  $p > 1$  such that  $f(n) = \log_p n$  for all  $n$ .