

27-th Polish Mathematical Olympiad 1975/76

Third Round

First Day

1. Is the number $\sin \frac{\pi}{18} \sin \frac{3\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18} \sin \frac{9\pi}{18}$ rational?
2. Four sequences of real numbers $(a_n), (b_n), (c_n), (d_n)$ satisfy for all n ,

$$\begin{aligned} a_{n+1} &= a_n + b_n, & b_{n+1} &= b_n + c_n, \\ c_{n+1} &= c_n + d_n, & d_{n+1} &= d_n + a_n. \end{aligned}$$

Prove that if $a_{k+m} = a_m, b_{k+m} = b_m, c_{k+m} = c_m, d_{k+m} = d_m$ for some $k \geq 1, n \geq 1$, then $a_2 = b_2 = c_2 = d_2 = 0$.

3. Prove that for each tetrahedron, the three products of pairs of opposite edges are sides of a triangle.

Second Day

4. The diagonals of some quadrilateral with sides a, b, c, d are perpendicular. Prove that the diagonals of any other quadrilateral with sides a, b, c, d also are perpendicular.
5. A trawler is about to fish in territorial waters of a neighboring country, for what he has no licence. Whenever he throws the net, the coast-guard may stop him with the probability $1/k$, where k is a fixed positive integer. Each throw brings him a fish landing of a fixed weight. However, if the coast-guard stops him, they will confiscate his entire fish landing and demand him to leave the country. The trawler plans to throw the net n times before he returns to territorial waters in his country. Find n for which his expected profit is maximal.
6. An increasing function $f : \mathbb{N} \rightarrow \mathbb{R}$ satisfies

$$f(kl) = f(k) + f(l) \quad \text{for all } k, l \in \mathbb{N}.$$

Show that there is a real number $p > 1$ such that $f(n) = \log_p n$ for all n .