26-th Polish Mathematical Olympiad 1974/75

Third Round

First Day

1. A sequence $(a_k)_{k=1}^{\infty}$ has the property that there is a natural number n such that $a_1+a_2+\cdots+a_n=0$ and $a_{n+k}=a_k$ for all k. Prove that there exists a natural number N such that
$$\sum_{i=N}^{N+k} a_i \geq 0 \quad \text{for } k=0,1,2,\dots.$$

2. On the surface of a regular tetrahedron of edge length 1 are given finitely many segments such that every two vertices of the tetrahedron can be joined by a polygonal line consisting of given segments. Can the sum of the lengths of the given segments be less than $1+\sqrt{3}$?

3. Find the smallest positive number α for which there is a positive number β such that for all $0 \leq x \leq 1$,
$$\sqrt{1+x}+\sqrt{1-x} \leq 2-\frac{x^{\alpha}}{\beta}.$$
For each such α determine the smallest $\beta>0$ for which this condition holds.

Second Day

4. All decimal digits of some natural number are 1,3,7, and 9. Prove that one can rearrange its digits so as to obtain a number divisible by 7.

5. Show that it is possible to circumscribe a circle of radius R about, and inscribe a circle of radius r in some triangle with one angle equal to α, if and only if
$$\frac{2R}{r} \geq \frac{1}{\sin\frac{\alpha}{2}\left(1-\sin\frac{\alpha}{2}\right)}.$$

6. On the interval $[0,1]$ are given functions $S(x)=1-x$ and $T(x)=x/2$. Does there exist a function of the form $f=g_1\circ g_2\circ\cdots\circ g_n$, where $n\in\mathbb{N}$ and each g_k is either $S(x)$ or $T(x)$, such that
$$f\left(\frac{1}{2}\right)=\frac{1975}{2^{1975}}\,?$$

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