23-rd Polish Mathematical Olympiad 1971/72

Third Round

First Day

1. Polynomials $u_i(x) = a_i x + b_i$ ($a_i, b_i \in \mathbb{R}$, $i = 1,2,3$) satisfy

$$u_1(x)^n + u_2(x)^n = u_3(x)^n \quad \text{for some integer } n \geq 2.$$

   Prove that there exist real numbers A, B, c_1, c_2, c_3 such that $u_i(x) = c_i(Ax + B)$ for $i = 1,2,3$.

2. On the plane are given $n > 2$ points, no three of which are collinear. Prove that among all closed polygonal lines passing through these points, any one with the minimum length is non-selfintersecting.

3. Prove that there is a polynomial $P(x)$ with integer coefficients such that for all x in the interval $[\frac{1}{10}, \frac{9}{10}]$ we have $|P(x) - \frac{1}{2}| < \frac{1}{1000}$.

Second Day

4. Points A and B are given on a line having no common points with a sphere K. The feet P of the perpendicular from the center of K to the line AB is positioned between A and B, and the lengths of segments AP and BP both exceed the radius of K. Consider the set Z of all triangles ABC whose sides AC and BC are tangent to K. Prove that among all triangles in Z, a triangle T with a maximum perimeter also has a maximum area.

5. Prove that all subsets of a finite set can be arranged in a sequence in which every two successive subsets differ in exactly one element.

6. Prove that the sum of digits of the number 1972^n is not bounded from above when n tends to infinity.

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