21-st Polish Mathematical Olympiad 1969/70

Third Round

First Day

1. Diameter AB divides a circle into two semicircles. Points $P_1, P_2, \dots, P_n$ are given on one of the semicircles in this order. How should a point C be chosen on the other semicircle in order to maximize the sum of the areas of triangles $CP_1P_2, CP_2P_3, \dots, CP_{n-1}P_n$?

2. Consider three sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$, each of which has pairwise distinct terms. Prove that there exist two indices k and l for which $k < l$, $a_k < a_l$, $b_k < b_l$, and $c_k < c_l$.

3. Prove that an integer $n > 1$ is a prime number if and only if, for every integer k with $1 \le k \le n-1$, the binomial coefficient $\binom{n}{k}$ is divisible by n.

Second Day

4. In the plane are given two mutually perpendicular lines and n rectangles with sides parallel to the two lines. Show that if every two rectangles have a common point, then all the rectangles have a common point.

5. In how many ways can a set of 12 elements be partitioned into six two-element subsets?

6. Find the smallest real number A such that, for every quadratic polynomial $f(x)$ satisfying $|f(x)| \le 1$ for $0 \le x \le 1$, it holds that $f'(0) \le A$.

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