59-th Polish Mathematical Olympiad 2007/08

Third Round

Rzeszów, April 9–10, 2008

First Day

1. The numbers $1,2,\dots,n^2$ are arranged in the cells of an $n\times n$ board in such a way that the numbers $1,\dots,n$ are in the first row (in this order), $n+1,\dots,2n$ in the second, etc. We choose n cells of the board, no two of which are in the same row or column. Let a_i be the chosen number in the i-th row. Prove that
$$\frac{1^2}{a_1}+\frac{2^2}{a_2}+\cdots+\frac{n^2}{a_n}\geq\frac{n+2}{2}-\frac{1}{n^2+1}\,.$$

2. A function f in three real variables satisfies for any a,b,c,d,e the equality
$$f(a,b,c)+f(b,c,d)+f(c,d,e)+f(d,e,a)+f(e,a,b)=a+b+c+d+e.$$
Prove that for any real numbers $x_1,\dots,x_n$ ($n\geq 5$) it holds that
$$f(x_1,x_2,x_3)+(x_2,x_3,x_4)+\cdots+f(x_n,x_1,x_2)=x_1+x_2+\cdots+x_n.$$

3. A convex pentagon $ABCDE$ is such that $BC=DE$, $\angle ABE=\angle CAB=\angle AED-90^\circ$ and $\angle ACB=\angle ADE$. Prove that $BCDE$ is a parallelogram.

Second Day

4. Each point of a plane with integer coordinates is painted in white or black. Show that there exists an infinite and centrally symmetric subset of colored points whose points are of the same color.

5. The areas of all sections of a parallelepiped $\mathscr{R}$ by planes, passing through the midpoints of three pairwise disjoint and non-parallel edges, are equal. Show that R is a cuboid.

6. Let S be the set of positive integers which can be written in the form a^2+5b^2 for some co-prime integers a and b. Let p be a prime congruent to 3 modulo 4. Prove that if some integral multiple of p belongs to S, then $2p$ belongs to S as well.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com