58-th Polish Mathematical Olympiad 2006/07

Third Round

Stalowa Wola, April 18–19, 2007

First Day

1. In an acute-angled triangle ABC point O is the circumcenter, CD is the altitude, E a point on side AB, and M the midpoint of CE. The perpendicular to OM at M intersects the lines AC and BC at K and L respectively. Prove that $\frac{LM}{MK} = \frac{AD}{DB}$.

2. A positive integer is said to be *white* if it is equal to 1 or to a product of an even number of (not necessarily distinct) prime factors. Other positive integers are called *black*. Does there exist a positive integer whose sum of white divisors equals the sum of black divisors?

3. A plane is divided into unit squares. A positive integer should be written in each unit square so that each positive integer occurs exactly once. Decide whether this can be done in such a way that the number in each square divides the sum of the numbers in the four neighboring squares.

Second Day

4. Given an integer $n \geq 1$, find the number of possible values of the product km, where k and m are integers with $n^2 \leq k \leq m \leq (n+1)^2$.

5. A tetrahedron $ABCD$ is such that

$$\angle BAC + \angle BDC = \angle ABD + \angle ACD,$$
$$\angle BAD + \angle BCD = \angle ABC + \angle ADC.$$

Prove that the center of the circumscribed sphere of the tetrahedron lies on the line passing through the midpoints of AB and CD.

6. The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined by $a_0 = -1$ and

$$a_n + \frac{a_{n-1}}{2} + \frac{a_{n-2}}{3} + \cdots + \frac{a_1}{n} + \frac{a_0}{n+1} = 0 \quad \text{for } n \geq 1.$$

Show that $a_n > 0$ for $n \geq 1$.

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