52-nd Polish Mathematical Olympiad 2000/01

Final Round

April 3–4, 2001

First Day

1. Prove that for all nonnegative real numbers $x_1, x_2, \dots, x_n$ ($n \geq 2$) the following inequality holds:
$$\sum_{i=1}^{n} i x_i \leq \binom{n}{2} + \sum_{i=1}^{n} x_i^i.$$

2. Prove that, for any interior point P of a regular tetrahedron with edge 1, the sum of the distances from P to the vertices of the tetrahedron is not greater than 3.

3. Consider the sequence (x_n) defined by
$$x_1 = a,\ \ x_2 = b,\ \ x_{n+2} = x_{n+1} + x_n \text{ for } n = 1, 2, 3, \dots,$$
where a and b are real numbers. We call a number c a *multiple value* of sequence (x_n) if there exist positive integers $k \neq l$ such that $x_k = x_l = c$. Prove that there exist a and b for which (x_n) possesses more than 2000 different multiple values. Moreover, prove that (x_n) cannot have infinitely many different multiple values.

Second Day

4. Suppose that a and b are integers such that $2^n a + b$ is a perfect square for all $n \in \mathbb{N}$. Show that $a = 0$.

5. Points K and L are taken on the sides BC and CD of a parallelogram $ABCD$, respectively, such that $BK \cdot AD = DL \cdot AB$. The segments DK and BL meet at point P. Prove that $\angle DAP = \angle BAC$.

6. Let $n_1 < n_2 < \cdots < n_{2000} < 10^{100}$ be given positive integers. Show that there exist two nonempty and disjoint subsets A and B of $\{n_1, n_2, \dots, n_{2000}\}$ such that:
   (i) A and B have the same number of elements;
   (ii) A and B have the same sums of elements;
   (iii) A and B have the same sums of the squares of elements.

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