51-st Polish Mathematical Olympiad 1999/2000

Final Round

April 3–4, 2000

First Day

1. For a given integer $n \geq 2$, find the number of nonnegative real solutions of the systeming system of equations:

$$\begin{cases} x_2 + x_1^2 & = 4x_1 \\ x_3 + x_2^2 & = 4x_2 \\ \quad \cdots\cdots & \cdots \\ x_1 + x_n^2 & = 4x_n. \end{cases}$$

2. Point P is taken in the interior of a triangle ABC with $AC = BC$ such that $\angle PAB = \angle PBC$. Point M is the midpoint of AB. Prove that

$$\angle APM + \angle BPC = 180^\circ.$$

3. The sequence (p_n) of natural numbers satisfies:

   (i) p_1 and p_2 are primes;

   (ii) For $n \geq 3$ the number p_n is the greatest proper divisor of $p_{n-1} + p_{n-2} + 2000$.

   Prove that the sequence (p_n) is bounded.

Second Day

4. In a regular pyramid with top vertex S and base $A_1A_2\dots A_n$ each lateral edge forms an angle of 60° with the base of the pyramid. For each $n \geq 3$ prove or disprove that there exist points $B_2, B_3, \dots, B_n$ lying on the edges $A_2S, A_3S, \dots, A_nS$, respectively, such that

$$A_1B_2 + B_2B_3 + \cdots + B_{n-1}B_n + B_nA_1 < 2A_1S.$$

5. Given an integer $n \geq 2$, find the smallest number k with the following property: From each set of k squares of an $n \times n$ chessboard one can choose a subset such that each row and column of the chessboard contains an even number of squares from this subset.

6. Suppose that $P(x)$ is a polynomial of an odd degree satisfying

$$P(x^2 - 1) = P(x)^2 - 1 \quad \text{for all } x.$$

   Prove that $P(x) = x$ for all x.

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