47-th Polish Mathematical Olympiad 1995/96

Second Round

February 23–24, 1996

First Day

1. Can every polynomial with integer coefficients be expressed as a sum of cubes of polynomials with integer coefficients?

2. A circle with center O is tangent to the sides AB, BC, CD, DA of a convex quadrilateral $ABCD$ at K, L, M, N. The lines KL and MN intersect at point S. Prove that $BD \perp OS$.

3. Prove that if $a, b, c \geq -\frac{3}{4}$ and $a+b+c=1$, then
$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1} \leq \frac{9}{10}.$$

Second Day

4. Let $a_1, a_2, \dots, a_{99}$ be a sequence of digits from the set $\{0, \dots, 9\}$ such that if for some $n \in \mathbb{N}$ $a_n = 1$, then $a_{n+1} \neq 2$, and if $a_n = 3$ then $a_{n+1} \neq 4$. Prove that there exist indices $k, l \in \{1, \dots, 98\}$ such that $a_k = a_l$ and $a_{k+1} = a_{l+1}$.

5. Find all integers x, y such that $x^2(y-1)+y^2(x-1)=1$.

6. Prove that every interior point of a parallelepiped with edges a, b, c is on the distance at most $\frac{1}{2}\sqrt{a^2+b^2+c^2}$ from some vertex of the parallelepiped.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com