44-th Polish Mathematical Olympiad 1992/93

Second Round

February 1993

First Day

1. If x,y,u,v are positive real numbers, prove the inequality

$$\frac{xu+xv+yu+yv}{x+y+u+v} \geq \frac{xy}{x+y} + \frac{uv}{u+v}.$$

2. Let be given a circle with center O and a point P outside the circle. A line l passes through P and cuts the circle at A and B. Let C be the point symmetric to A with respect to OP, and let m be the line BC. Prove that all lines m have a common point as l varies.

3. A tetrahedron $OA_1B_1C_1$ is given. Let $A_2,A_3 \in OA_1$, $A_2,A_3 \in OA_1$, $A_2,A_3 \in OA_1$ be points such that the planes $A_1B_1C_1, A_2B_2C_2$ and $A_3B_3C_3$ are parallel and $OA_1 > OA_2 > OA_3 > 0$. Let V_i be the volume of the tetrahedron $OA_iB_iC_i$ $(i = 1,2,3)$ and V be the volume of $OA_1B_2C_3$. Prove that $V_1 + V_2 + V_3 \geq 3V$.

Second Day

4. Let (x_n) be the sequence of positive integers such that $x_1 = 1$ and $x_n < x_{n+1} \leq 2n$ for each $n \in \mathbb{N}$. Show that for every positive integer k there exist indices r,s such that $x_r - x_s = k$.

5. Let D,E,F be points on the sides BC,CA,AB of a triangle ABC, respectively. Suppose that the inradii of the triangles AEF, BFD, CDE are all equal to r_1. If r_2 and r are the inradii of triangles DEF and ABC respectively, prove that $r_1 + r_2 = r$.

6. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditions $f(1000) = 999$ and $f(x)f(f(x)) = 1$ for all real x. Determine $f(500)$.

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