58-th Polish Mathematical Olympiad 2006/07

Second Round

February 23–24, 2007

First Day

1. A polynomial $P(x)$ has integer coefficients. Prove that if the polynomials $P(x)$ and $P(P(P(x)))$ have a common zero, then they also have a common integer zero.

2. Consider a convex pentagon $ABCDE$ with $BC = CD$, $DE = EA$ and $\angle BCD = \angle DEA = 90^\circ$. Prove that AC, CE and EB are sides of a triangle and find the angles of this triangle, knowing that $\angle ACE = \alpha$ and $\angle BEC = \beta$.

3. An equilateral triangle of side n is composed of n^2 equilateral triangular tiles of side 1. Each tile has one side white and the other side black. An allowed move is as follows: Choose a tile P having a common side with at least two other tiles whose top face is of different color than that of P; then turn P over. For each $n \geq 2$ determine whether there is an initial position permitting infinitely many such moves.

Second Day

4. Prove that if a, b, c, d are positive integers satisfying $ad = b^2 + bc + c^2$, then the number $a^2 + b^2 + c^2 + d^2$ is composite.

5. A convex quadrilateral $ABCD$ with $AB \neq CD$ is inscribed in a circle. Let $AKDL$ and $CMBN$ be rhombuses with side length a. Prove that the points K, L, M, N lie on a circle.

6. Positive numbers a, b, c, d satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 4$. Prove that

$$\sqrt[3]{\frac{a^3+b^3}{2}} + \sqrt[3]{\frac{b^3+c^3}{2}} + \sqrt[3]{\frac{c^3+d^3}{2}} + \sqrt[3]{\frac{d^3+a^3}{2}} + \leq 2(a+b+c+d) - 4.$$

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