57-th Polish Mathematical Olympiad 2005/06

Second Round

February 24–25, 2006

First Day

1. Positive integers a,b,c and x,y,z satisfy $|x-a| \le 1$, $|y-b| \le 1$, and
$$a^2+b^2=c^2, \quad x^2+y^2=z^2.$$
Prove that the sets $\{a,b\}$ and $\{x,y\}$ coincide.

2. In a triangle ABC with $AC+BC=3AB$, the incircle is centered at I and touches BC at D and AC at E. Let K and L be the points symmetric to D and E with respect to I. Prove that the points A,B,K,L lie on a circle.

3. Positive numbers a,b,c satisfy the condition $ab+bc+ca=abc$. Prove that
$$\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ca(c^3+a^3)} \ge 1.$$

Second Day

4. Given a natural number c, we define the sequence (a_n) by $a_1=1$ and
$$a_{n+1}=d(a_n)+c \quad \text{for } n=1,2,\dots,$$
where $d(m)$ denotes the number of positive divisors of $m \in \mathbb{N}$. Show that there is a positive integer k such that the sequence $a_k,a_{k+1},a_{k+2},\dots$ is periodic.

5. Let C be the midpoint of a segment AB. A circle o_1 passing through A and C and a circle o_2 passing through B and C intersect in two different points C and D. Point P is the midpoint of the arc AD of o_1 not containing C, and Q is that of the arc BD of o_2 not containing C. Prove that PQ is perpendicular to CD.

6. A prime number p and an integer n with $p \ge n \ge 3$ are given. A set A of sequences of length n with terms in the set $\{0,1,2,\dots,p-1\}$ has the following property: Any two sequences $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ from A differ in at least three positions. Find the largest possible cardinality of A.

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