56-th Polish Mathematical Olympiad 2004/05

Second Round

February 25–26, 2005

First Day

1. Find all positive integers n for which n^n+1 and $(2n)^{2n}+1$ are prime numbers.

2. In a convexs quadrilateral $ABCD$, point M is the midpoint of diagonal AC. Prove that if $\angle BAD = \angle BMC = \angle CMD$, then a circle can be inscribed in quadrilateral $ABCD$.

3. In space are given $n \geq 2$ points, no four of which are coplanar. Some of these points are connected by segments. Let K be the number of segments ($K > 1$) and T be the number of formed triangles. Prove that $9T^2 < 2K^3$.

Second Day

4. The polynomial $W(x) = x^2 + ax + b$ with integer coefficients has the following property: For every prime number p there is an integer k such that both $W(k)$ and $W(k+1)$ are divisible by p. Show that there is an integer m such that $W(m) = W(m+1) = 0$.

5. A rhombus $ABCD$ with $\angle BAD = 60^\circ$ is given. Points E on side AB and F on side AD are such that $\angle ECF = \angle ABD$. Lines CE and CF respectively meet line BD at P and Q. Prove that $\dfrac{PQ}{EF} = \dfrac{AB}{BD}$.

6. Prove that if real numbers a, b, c lie in the interval $[0, 1]$, then

$$\frac{a}{bc+1} + \frac{b}{ca+1} + \frac{c}{ab+1} \leq 2.$$

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