55-th Polish Mathematical Olympiad 2003/04

Second Round

February 20–21, 2004

First Day

1. Positive numbers a, b, c, d satisfy the equalities

$$\begin{aligned} a^3+b^3+c^3 &= 3d^3 \\ b^4+c^4+d^4 &= 3a^4 \\ c^5+d^5+a^5 &= 3b^5. \end{aligned}$$

   Prove that $a = b = c = d$.

2. In a convex hexagon $ABCDEF$ all sides have equal length and

$$\angle A + \angle C + \angle E = \angle B + \angle D + \angle F.$$

   Prove that the diagonals AD, BE, and CF are concurrent.

3. Determine all sequences $a_1, a_2, a_3, \dots$ of 1 and -1 that satisfy the equality $a_{mn} = a_m a_n$ for all m, n and have the property: Among any three successive terms a_n, a_{n+1}, a_{n+2}, both 1 and -1 occur.

Second Day

4. Find all positive integers n which have exactly $\sqrt{n}$ positive divisors.

5. Points D and E respectively are taken on the sides BC and CA of a triangle ABC such that $BD = AE$. Segments AD and BE meet at P. The bisector of $\angle ACB$ intersects segments AD and BE at Q and R respectively. Prove that $\dfrac{PQ}{AD} = \dfrac{PR}{BE}$.

6. There are $n \geq 5$ persons at a party. Assume that among any three of them some two know each other. Show that one can select at least $n/2$ of the persons and arrange them at a round table so that each person sits between two of his/her acquaintances.

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