53-rd Polish Mathematical Olympiad 2001/02

Second Round

February 22–23, 2002

First Day

1. Prove that all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$\forall x \in \mathbb{R} \;\; f(x) = f(2x) = f(1-x)$$
are periodic.

2. In a convex quadrilateral $ABCD$ the following equalities
$$\angle ADB = 2\angle ACB \text{ and } \angle BDC = 2\angle BAC$$
hold. Prove that $AD = CD$.

3. A positive integer n is given. In an association consisting of n members work 6 commissions. Each commission contains at least $n/4$ persons. Prove that there exists two commissions containing at least $n/30$ persons in common.

Second Day

4. Find all numbers $p \leq q \leq r$ such that all the numbers
$$pq + r,\ pq + r^2,\ qr + p,\ qr + p^2,\ rp + q,\ rp + q^2$$
are prime.

5. Triangle ABC with $\angle BAC = 90^\circ$ is the base of the pyramid $ABCD$. Moreover it holds
$$AD = BD \text{ and } AB = CD.$$
Prove that $\angle ACD \geq 30^\circ$.

6. Find all positive integers n such that for all real numbers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ the following inequality
$$x_1x_2\dots x_n + y_1y_2\dots y_n \leq \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2} \cdot \dots \cdot \sqrt{x_n^2 + y_n^2}$$
holds.

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